Using CSP refutation capabilities to refine AI-based Approximations

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"Uncertainty modeling and analysis with intervals"

Outline

Verifying programs with floating-point computations

Context

- Embedded Systems (transportation, nuclear energy...) rely more and more on floating-point computations
- C language is widely used for such applications
- Floats $\rightarrow$ an additional source of errors

Problems with floating-point numbers

- Counter intuitive Properties and “pitfalls” of Floating-point arithmetic:
  - Arithmetic operators are neither associative nor distributive
  - Reasoning with rounding, absorption, cancellation
- Examples (in simple precision)
  - Absorption : $10^7 + 0.5 = 10^7$
  - Cancellation : $((1 - 10^{-7}) - 1) * 10^7 = -1.192...(\neq 1)$
  - $(10000001 - 10^7) + 0.5 \neq 10000001 - (10^7 + 0.5)$
  - $0.1 = (0.000110011001100\ldots)$

Semantics of program with floating-point numbers

Programs are run on the floats but:

- Specification, properties of programs
  $\rightsquigarrow$ Users are reasoning with real numbers
- Programs are sometimes written with the semantics of real numbers "in mind"
- Differences between computations over real numbers and computations over the floats
  $\rightarrow$ reveal problems with floats

Abstract Interpretation

$\rightarrow$ Approximations of computations over floats and computations over the real numbers

Objective & Approach

- **Goal:** Refine the approximations of the domains of the program variables computed by abstract interpretation

- **Approach:** Use local consistencies to “shave” the domains
  - $\rightarrow$ More accurate “ semantics”
  - $\rightarrow$ Complementary to “abstract domains”

Example (Abstract Interpretation)

Abstract Interpretation requires:

1. A semantics to compute the states of a program at various checkpoints
2. An abstraction to represent the states of a program
3. A fixed point computation of the equations of the semantics

Example (abstract domain of intervals)

```
float x = [0,10];
float y = x*x - x;
if (y >= 0)
  y = x/10;
else
  y = x*x + 2;
y = 1 / (y-1.5);
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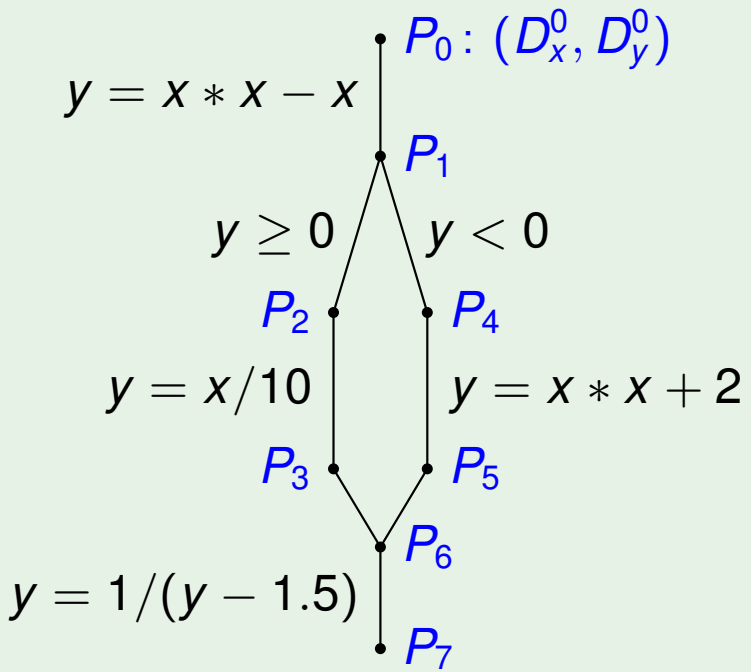

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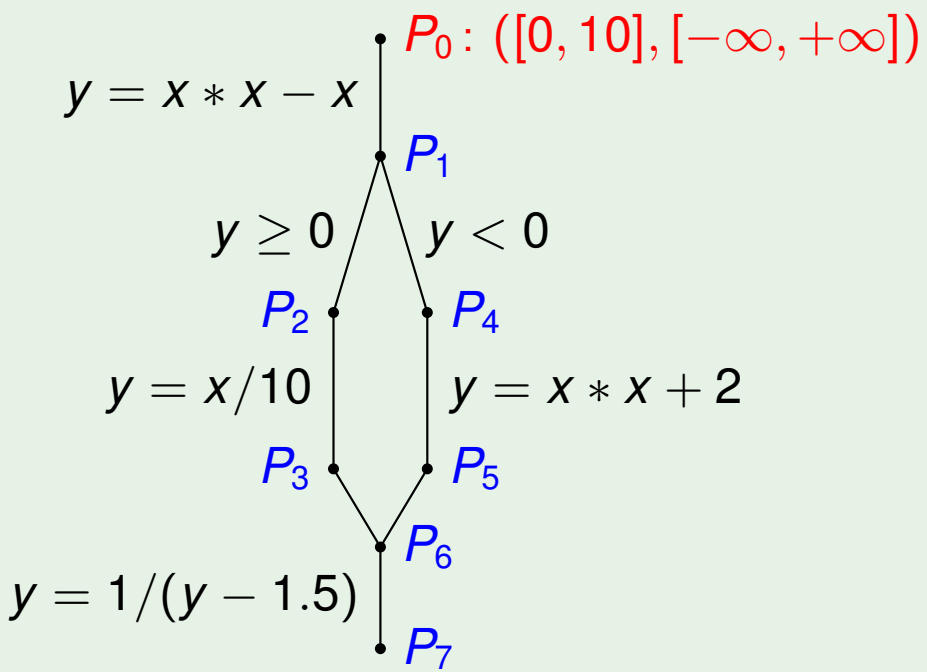

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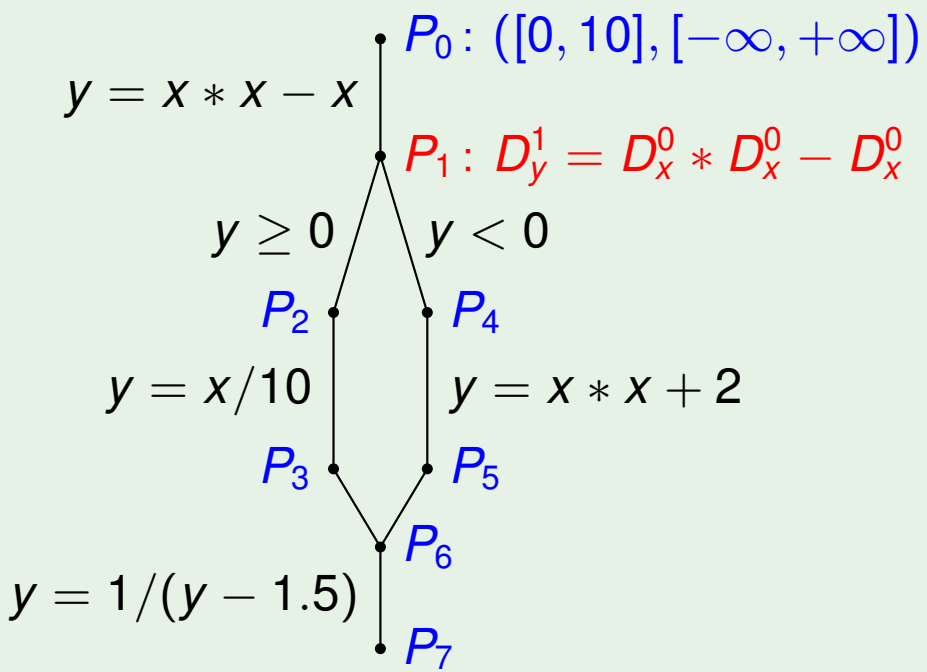

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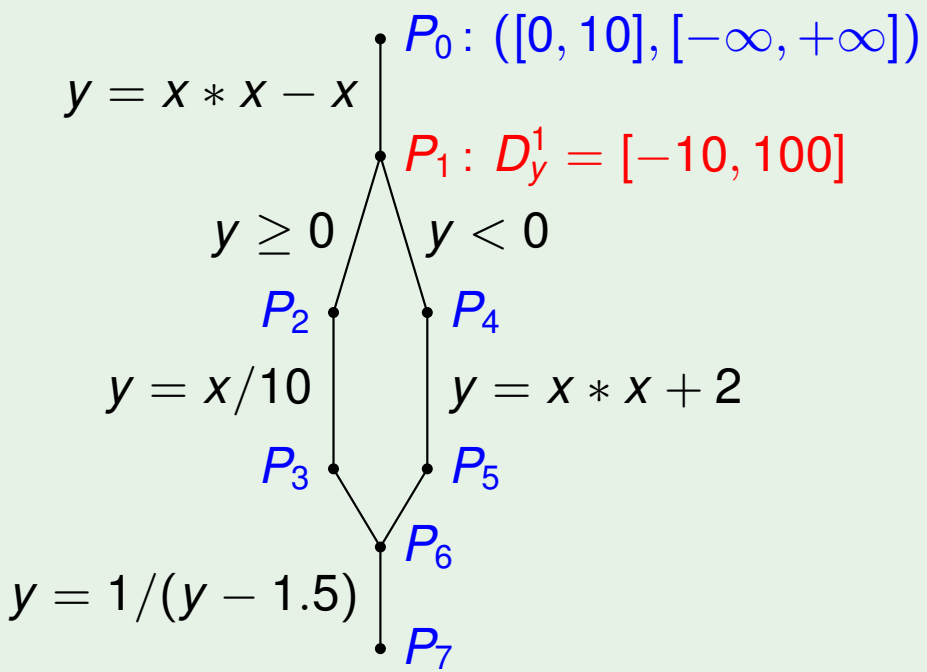

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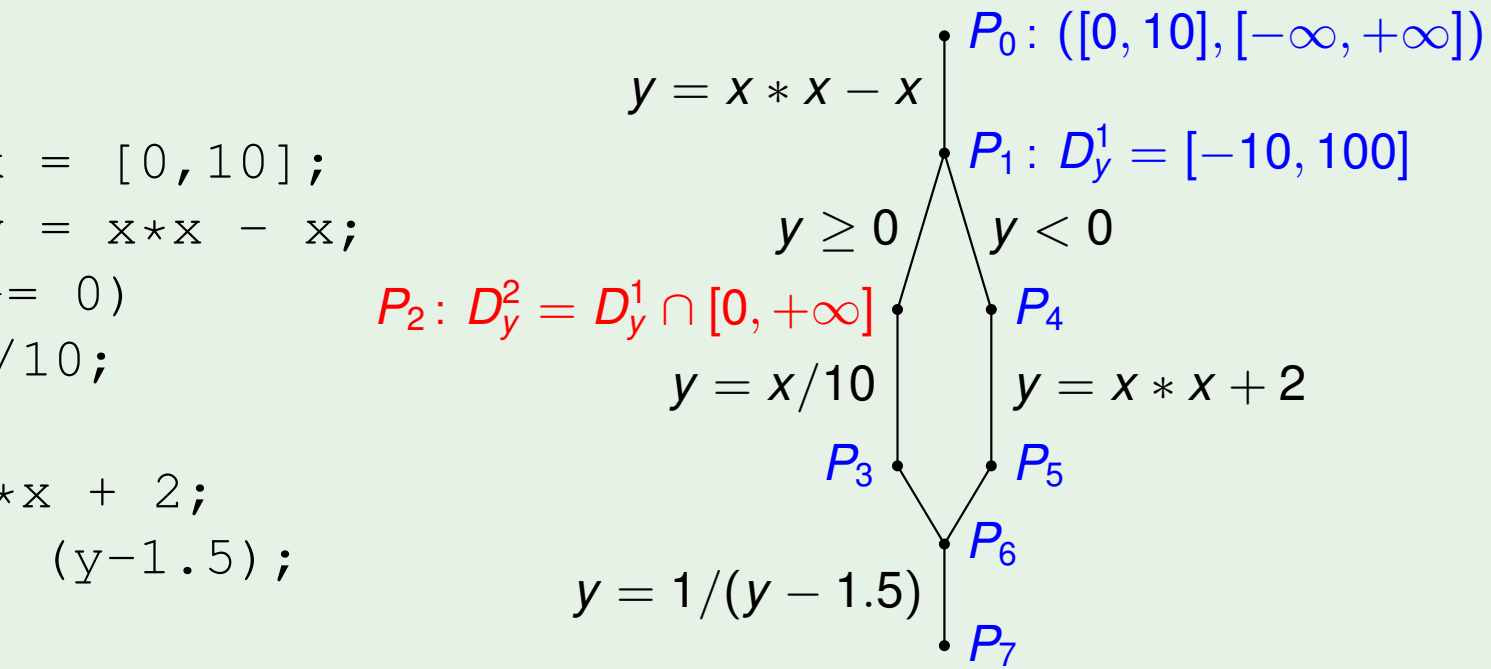

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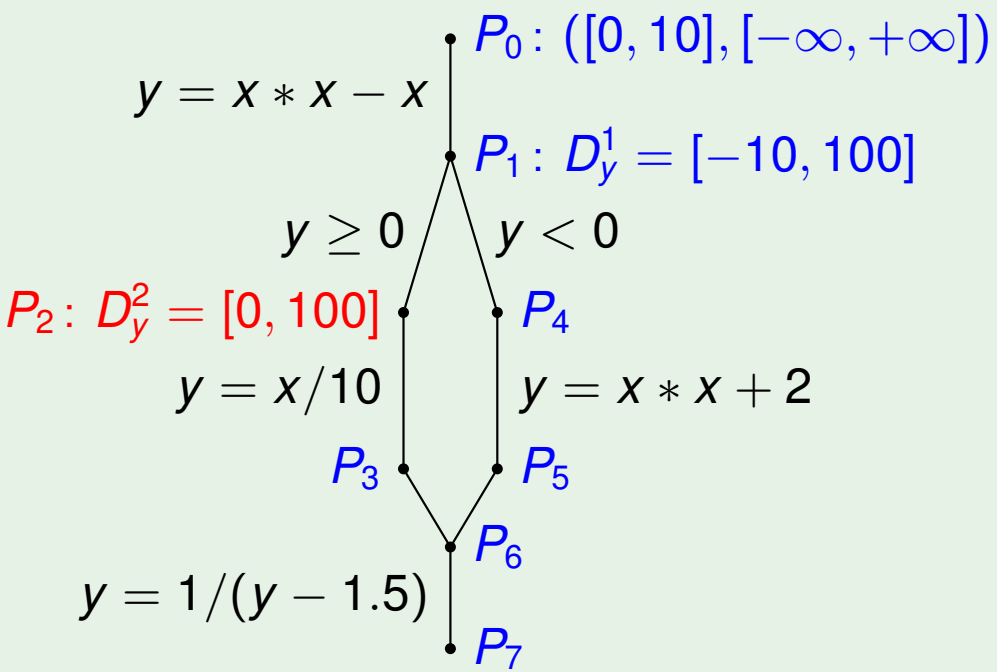

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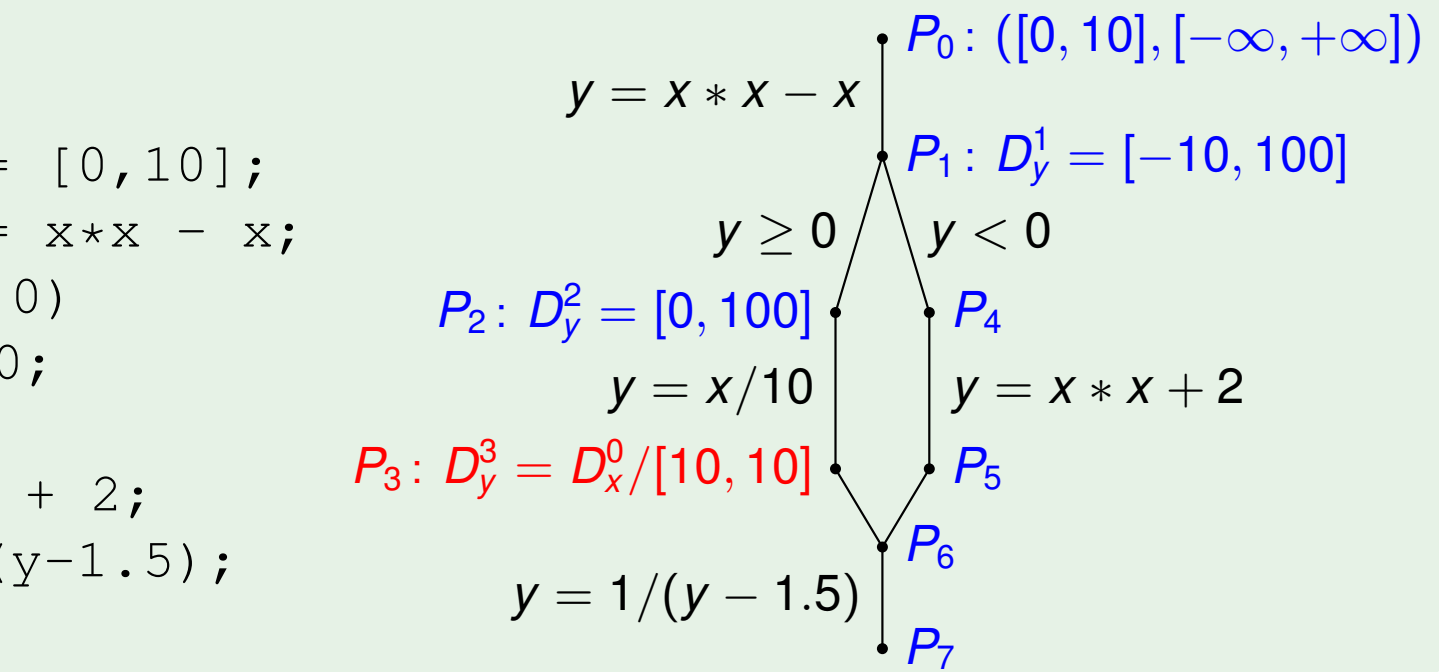

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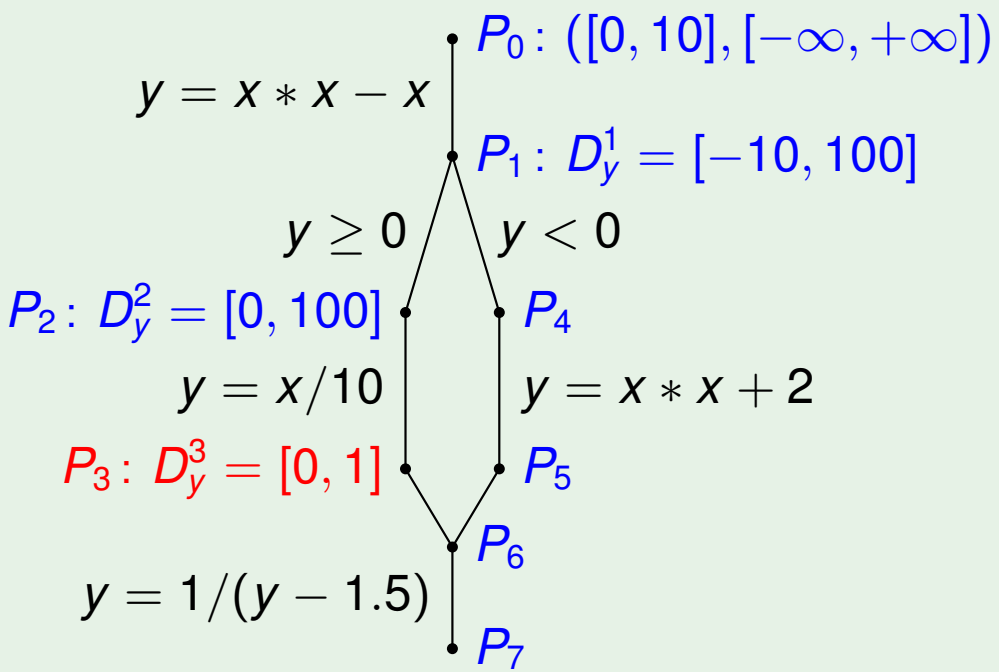

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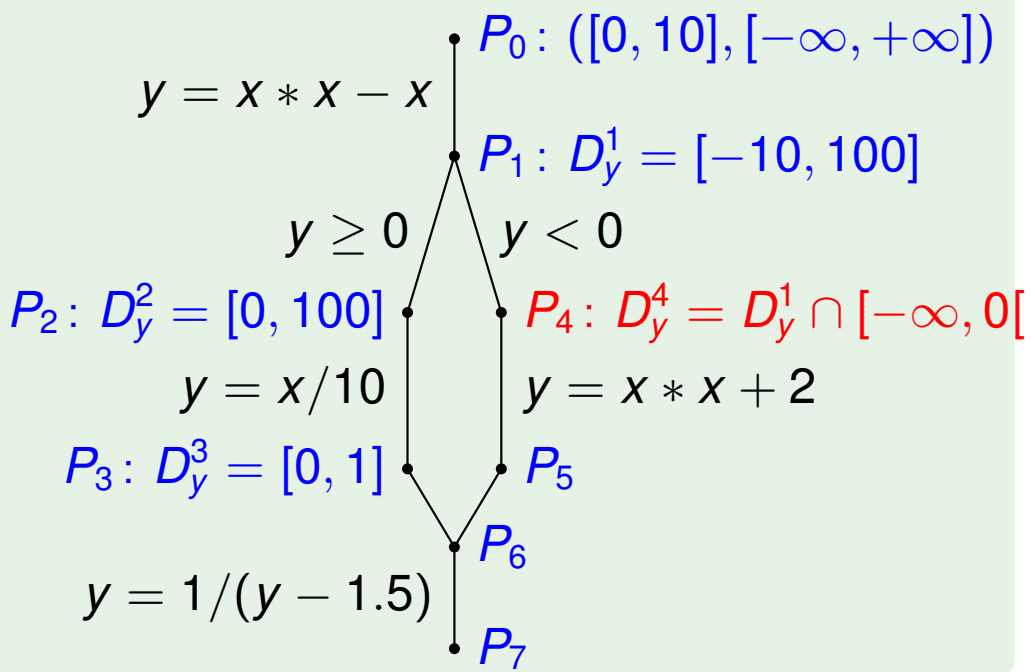

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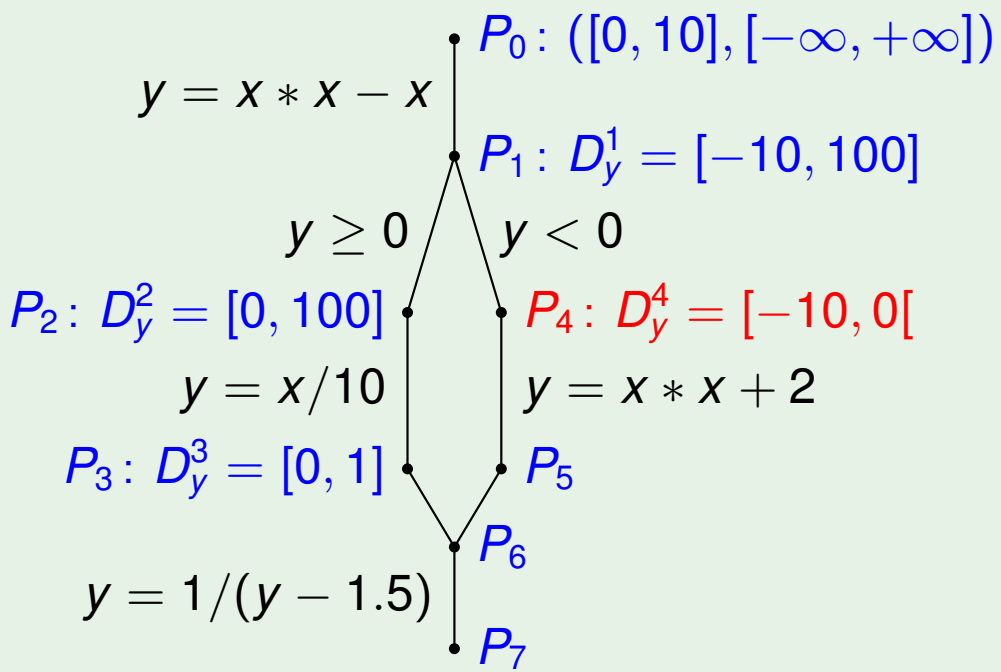

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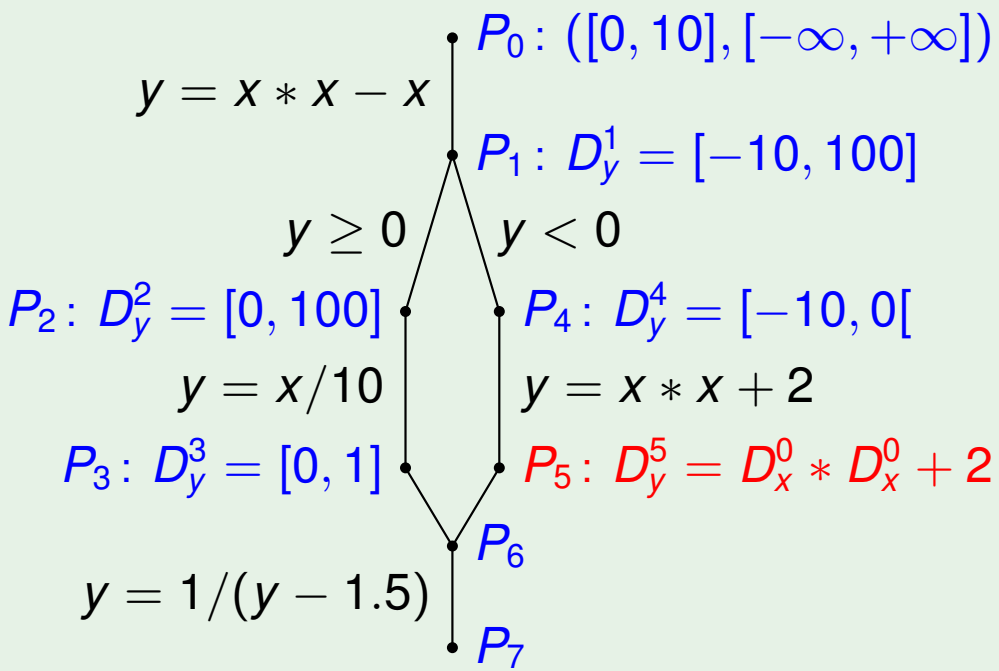

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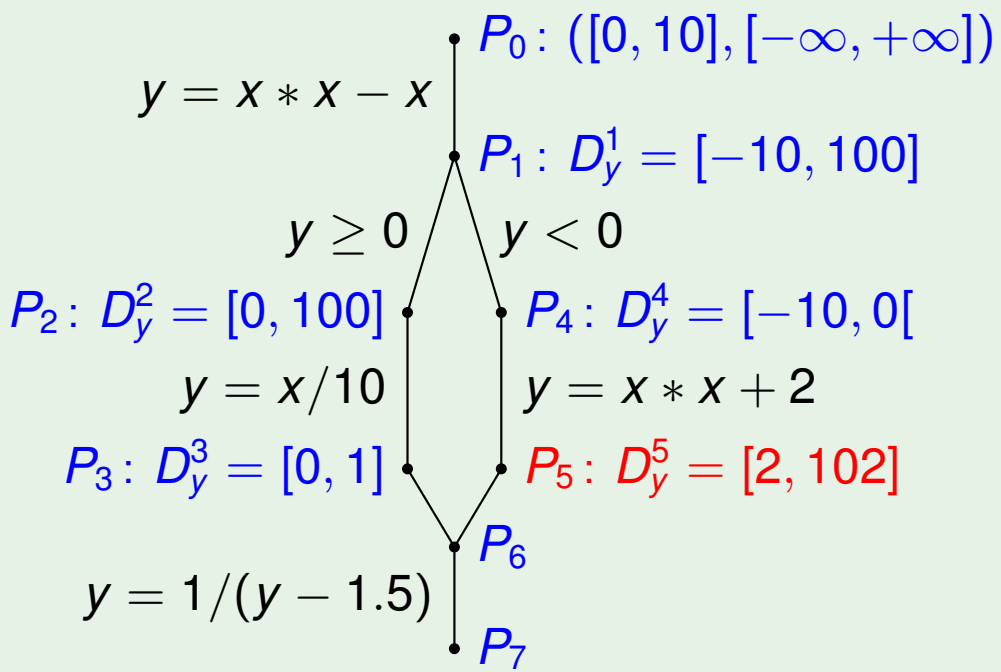

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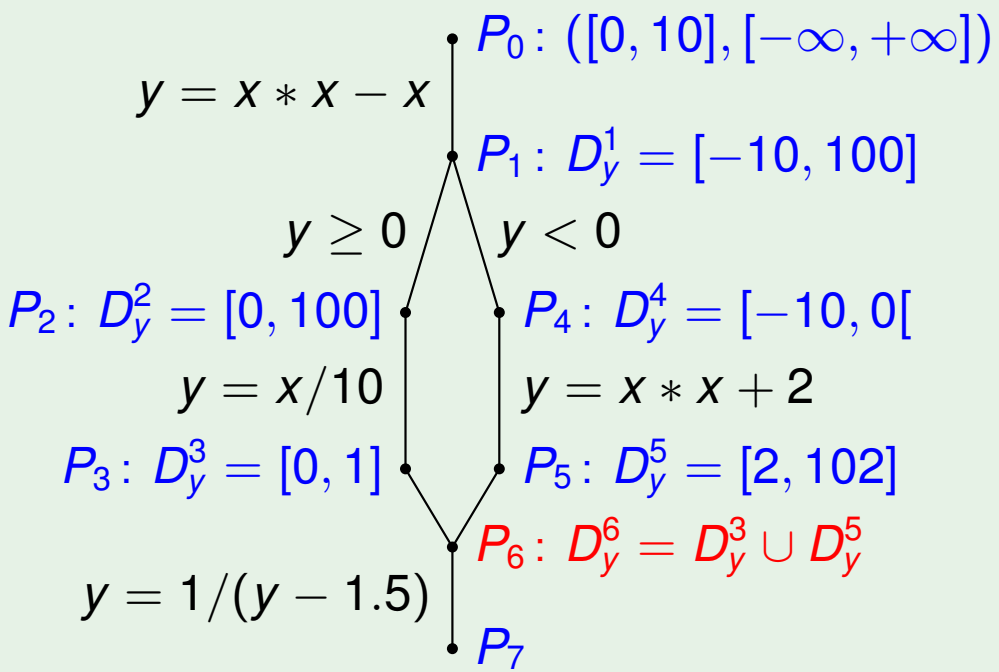

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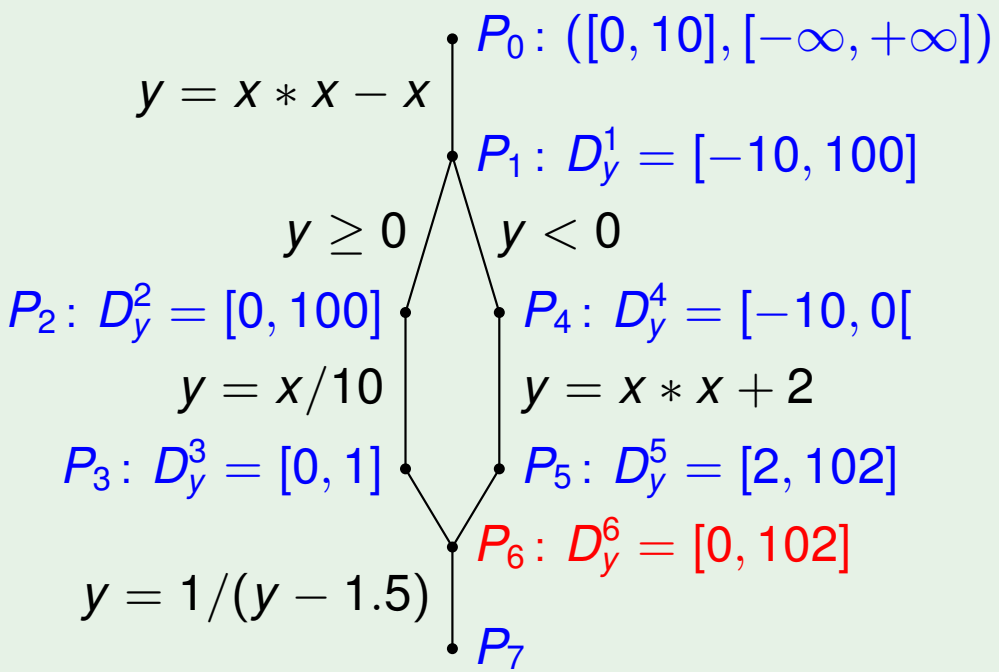

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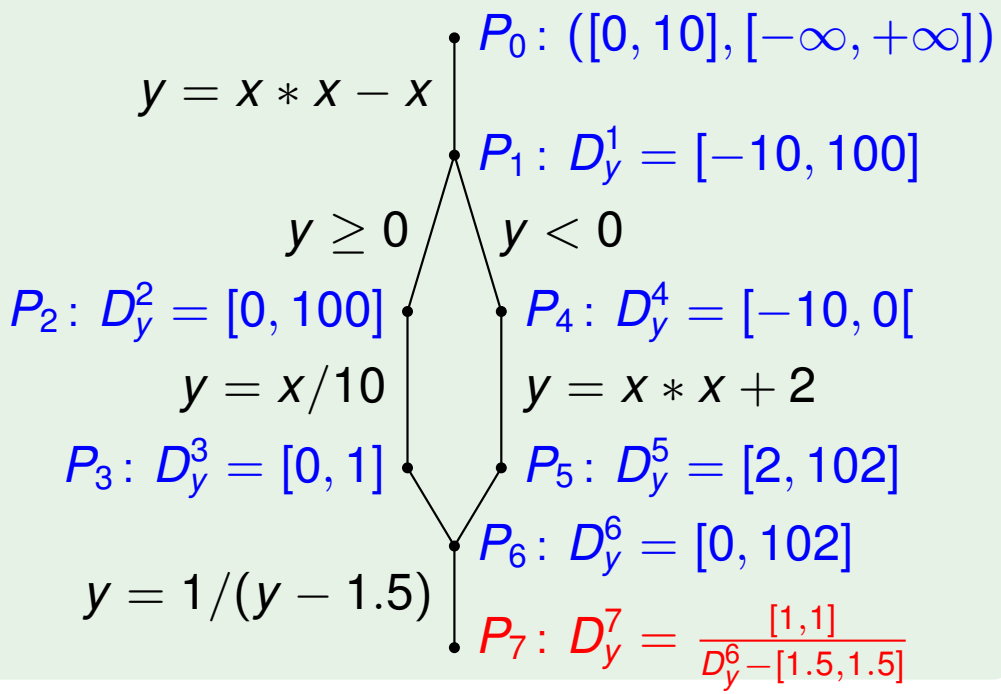

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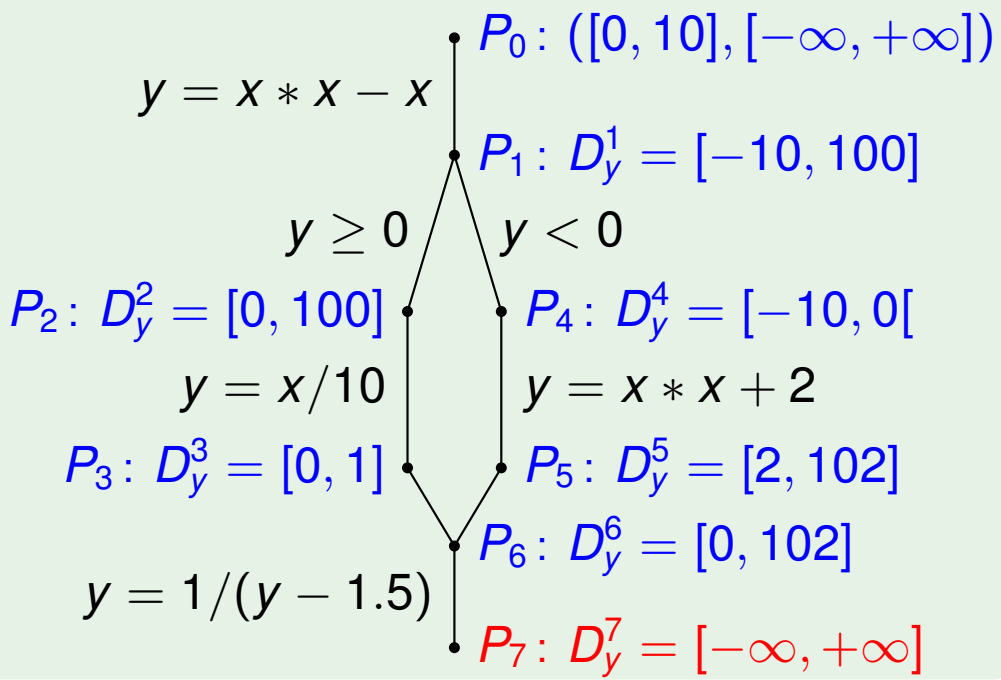

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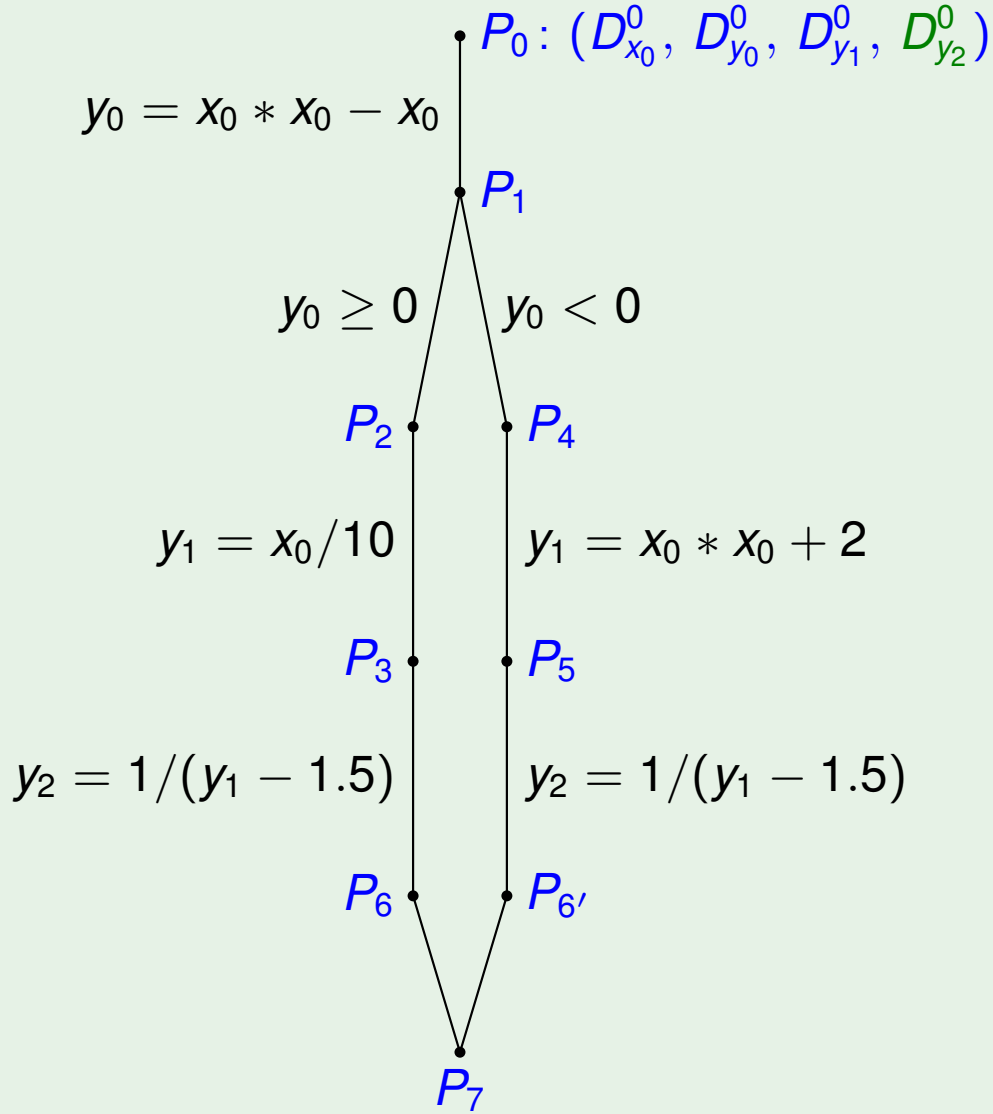

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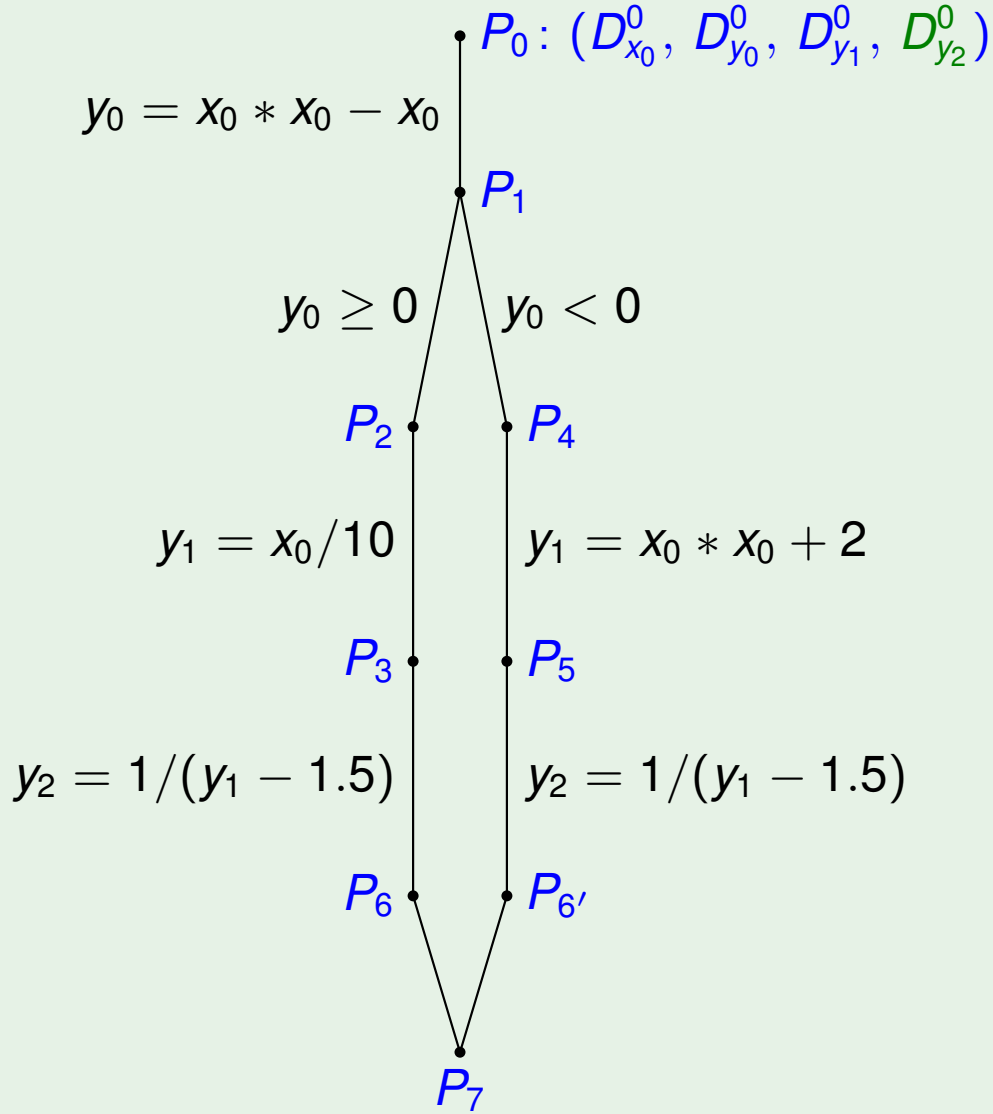

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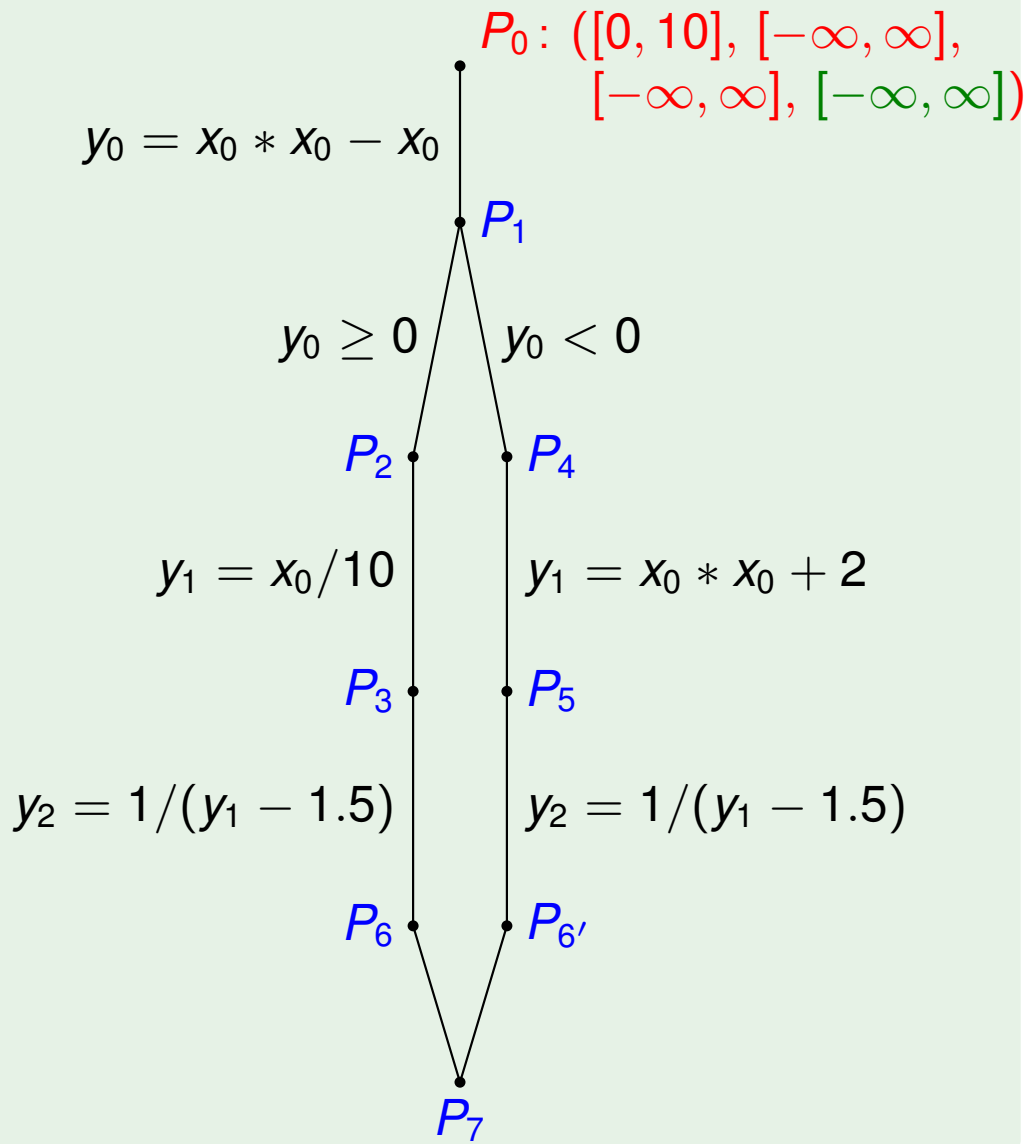

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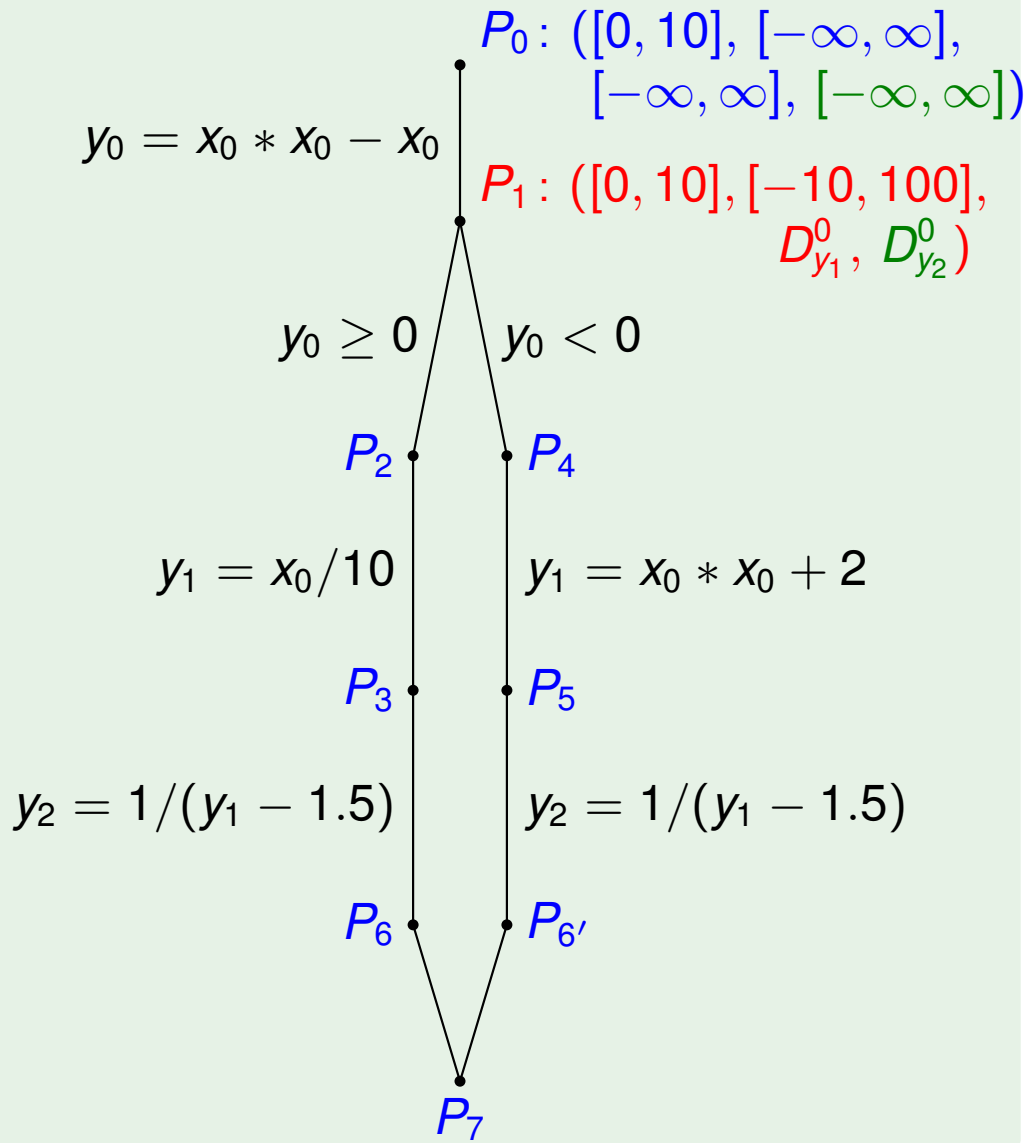

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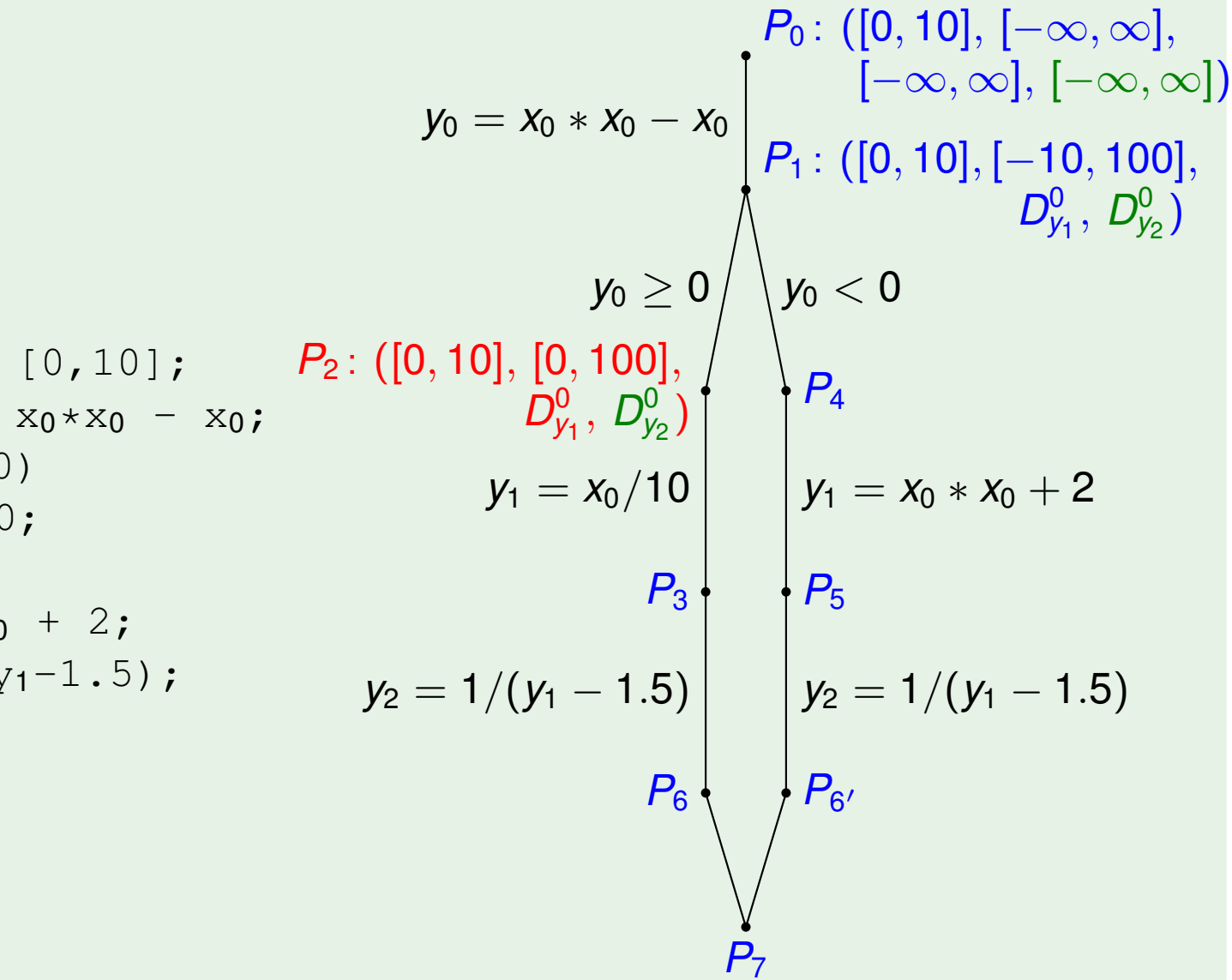

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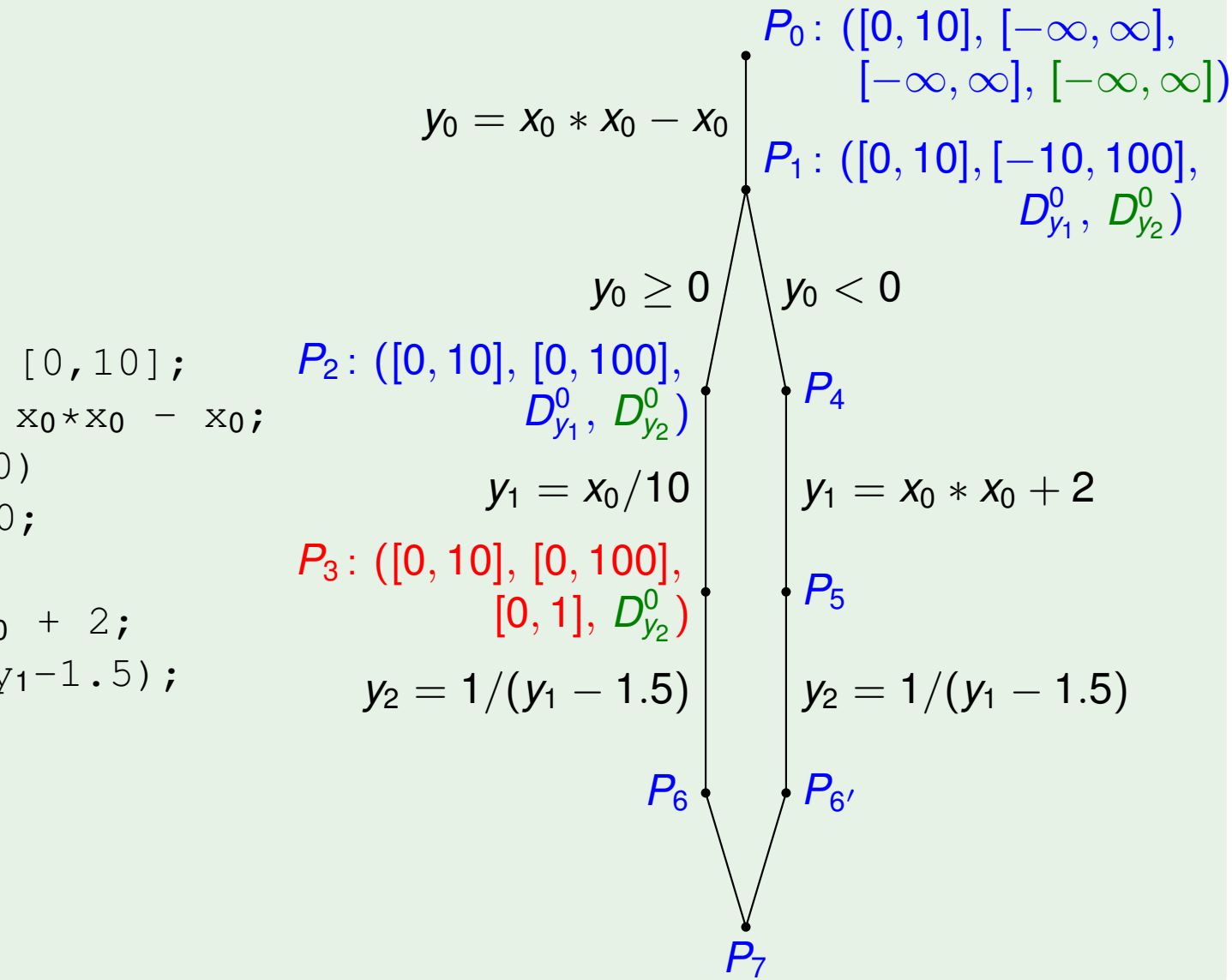

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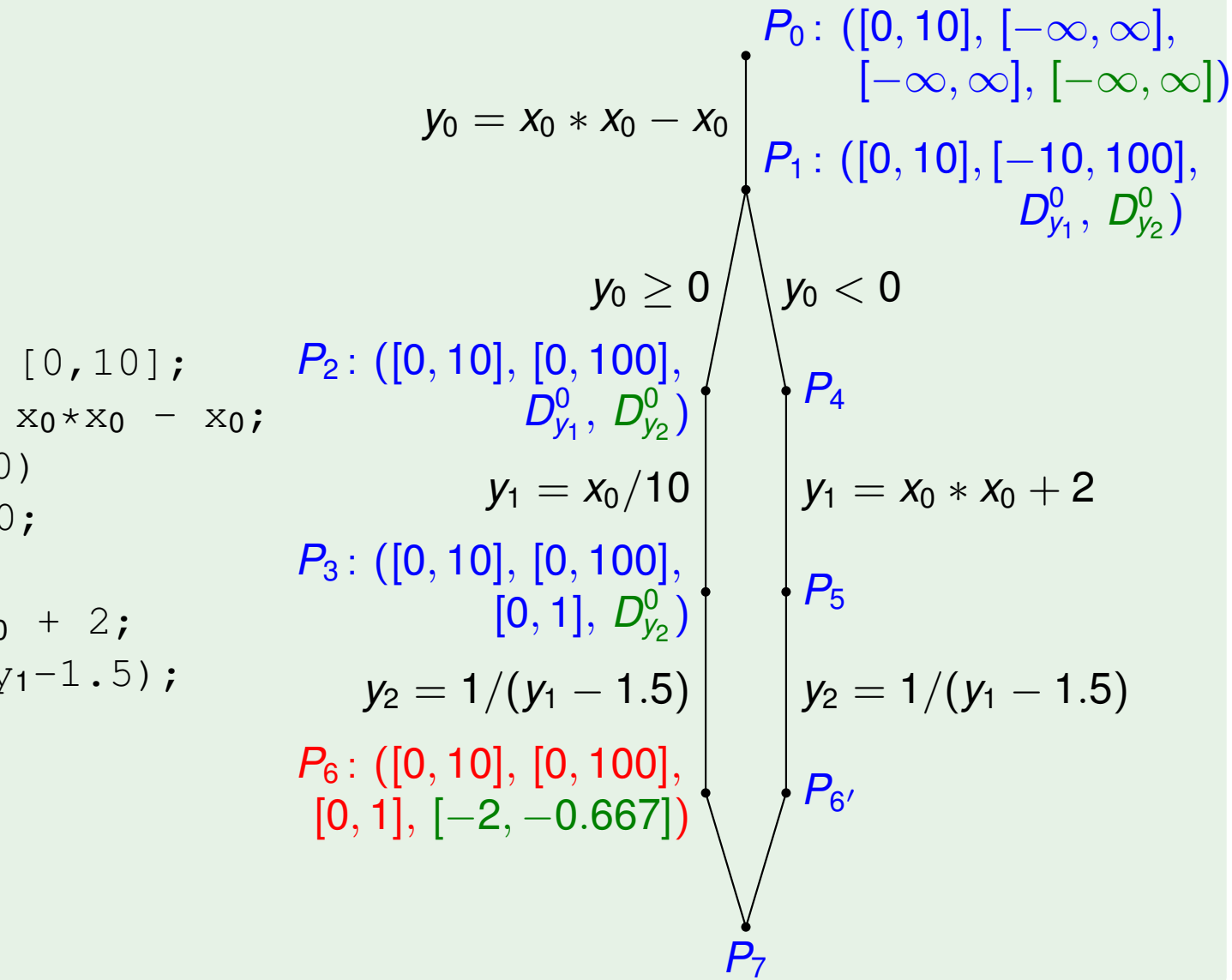

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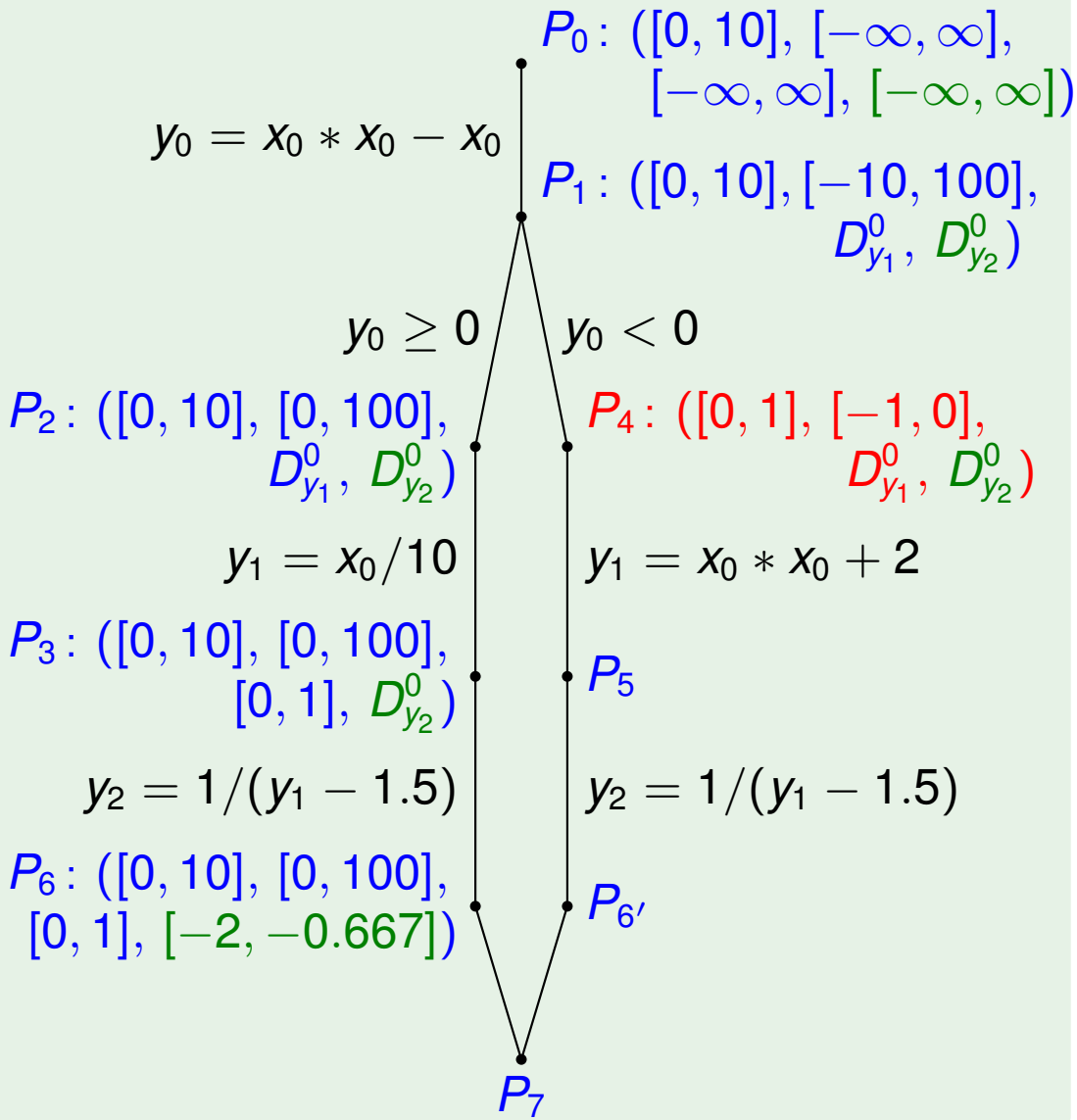

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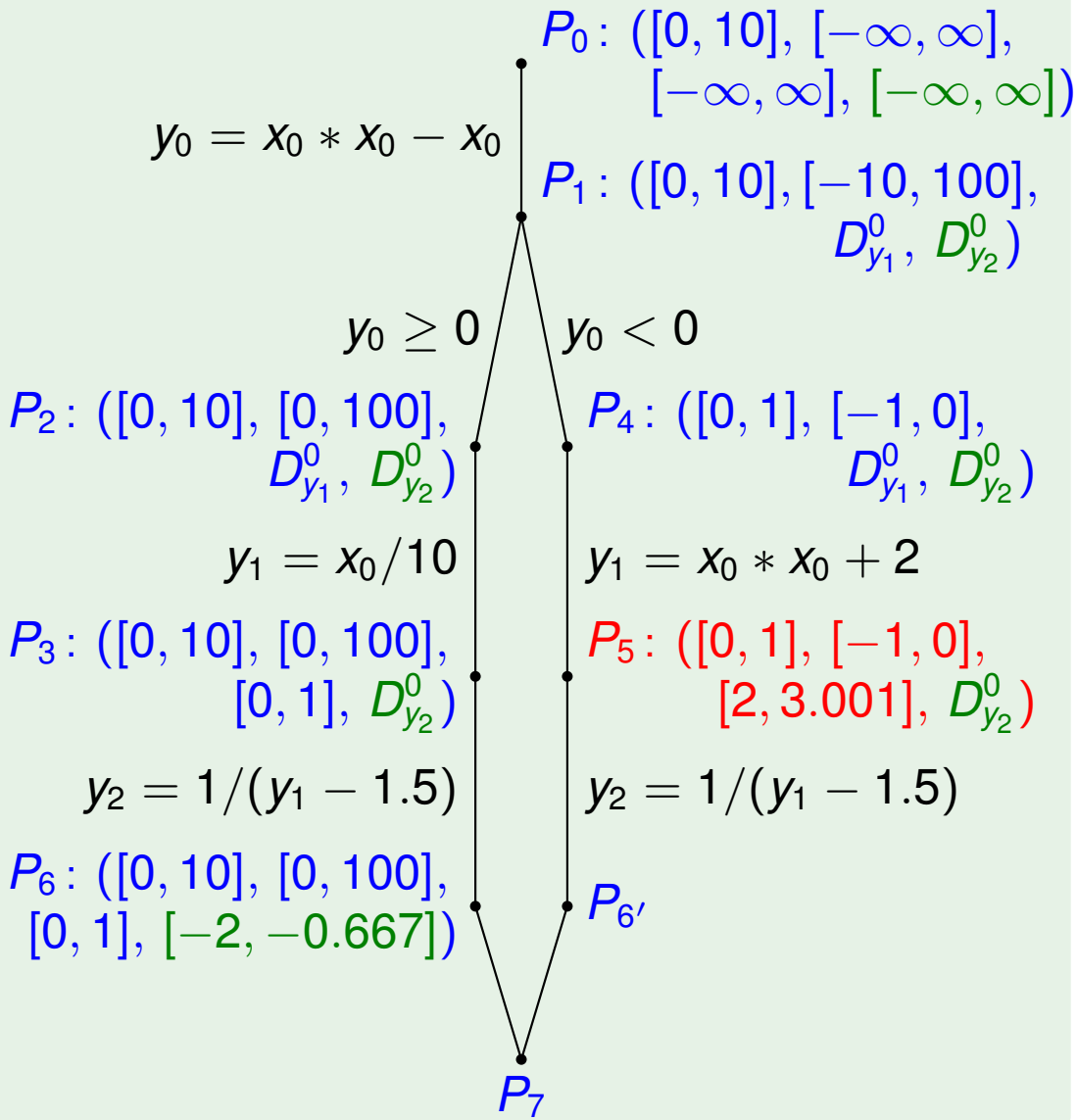

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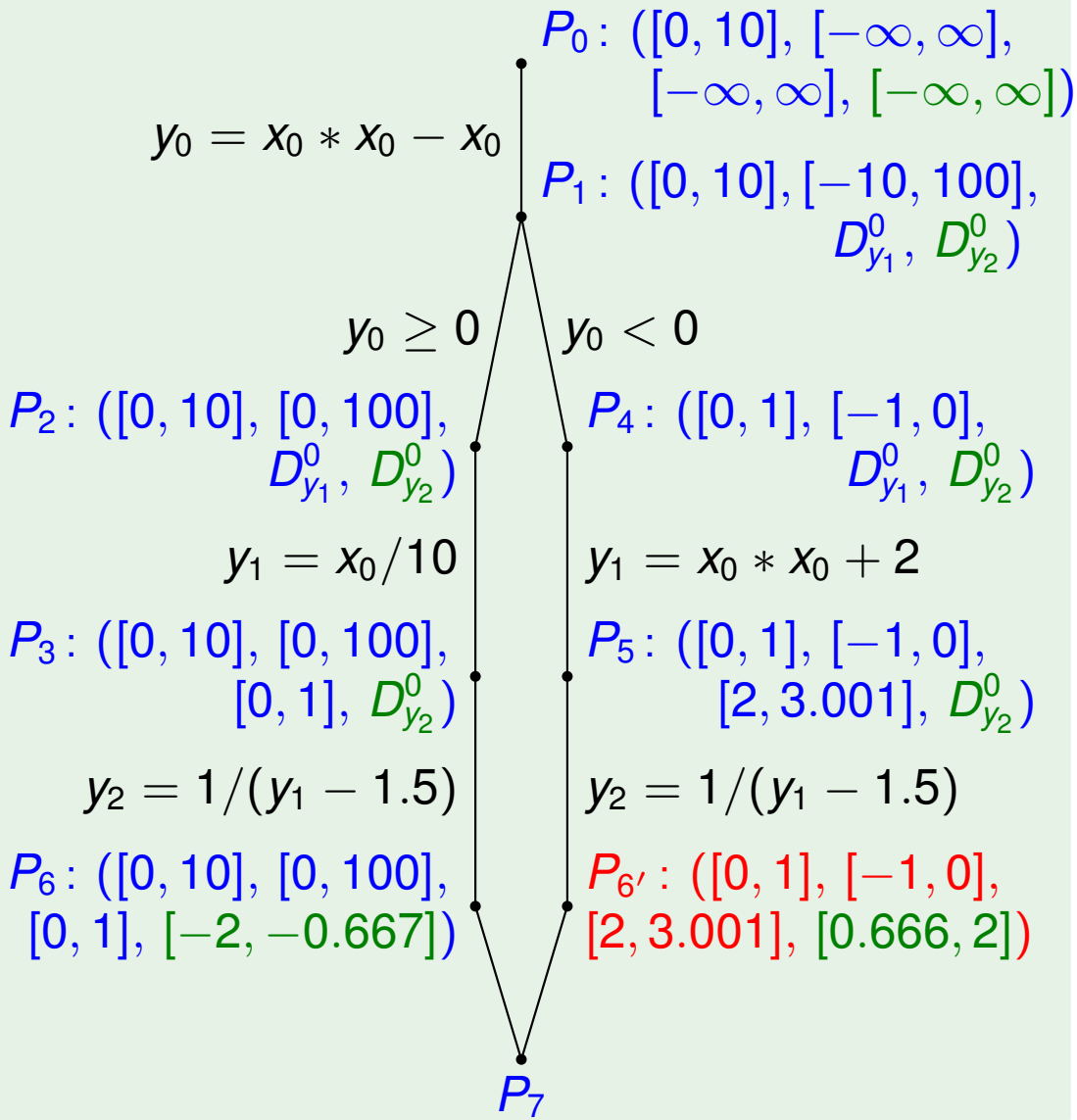

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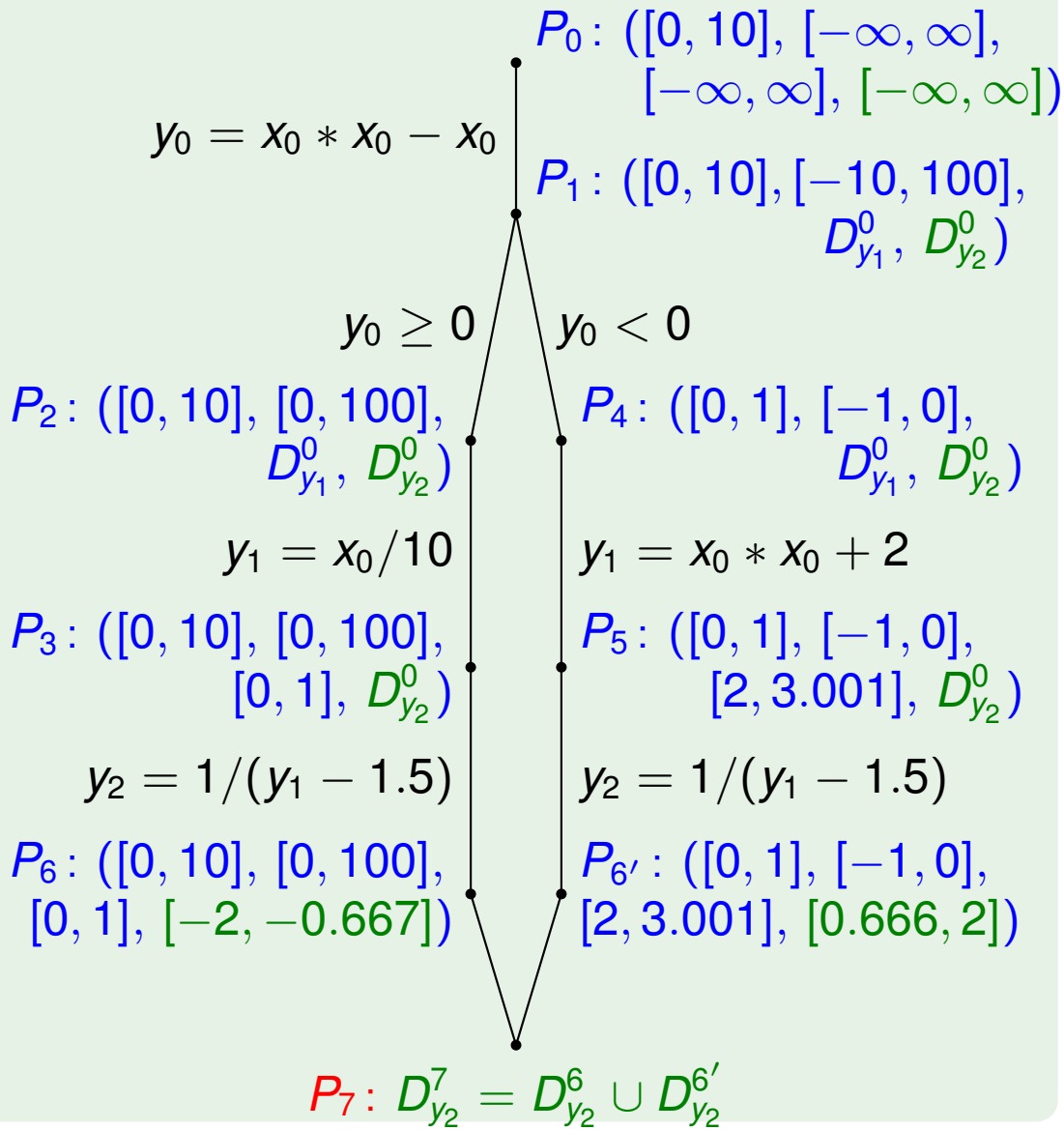

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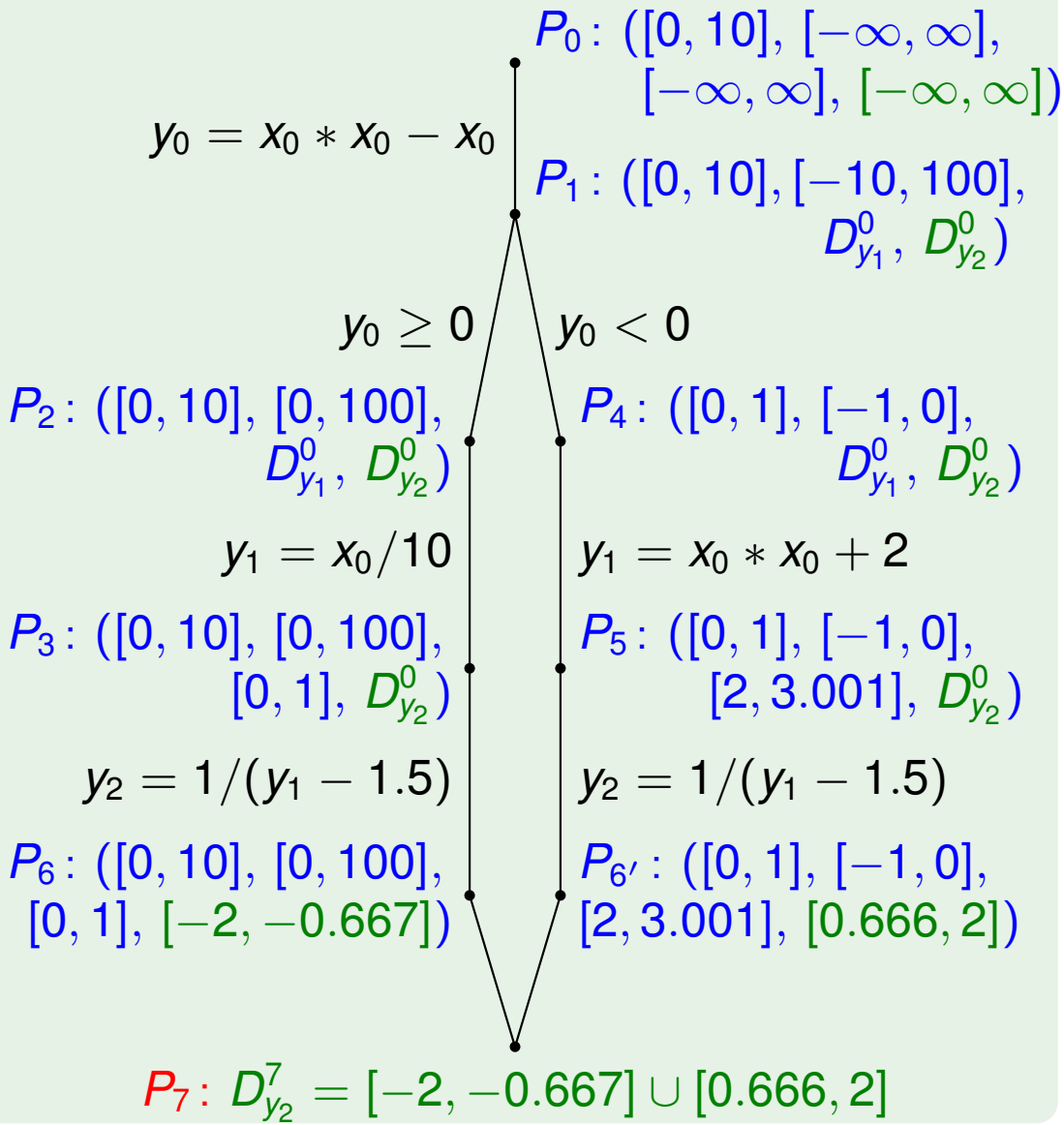

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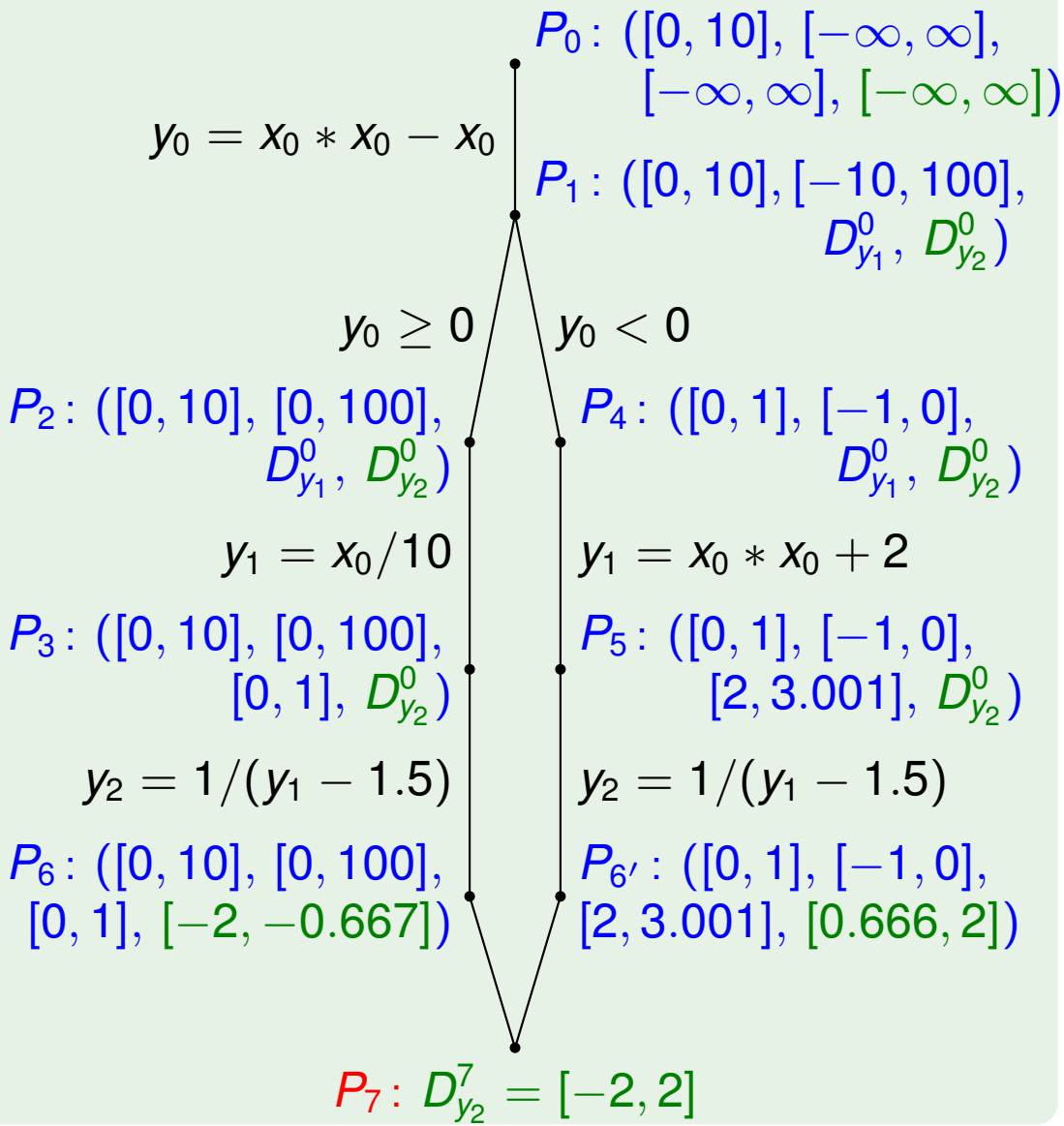


Abstract Interpretation, Fluctuat

Static analyzer of C programs using Zonotopes
$\rightarrow$ estimates rounding errors and their propagation

- Programs are considered both:
  - As a specification over the reals
  - As an implementation over the floats
- Fluctuat computes:
  - An over-approximation of the domain-bounds of each variable considered as a real number
  - An over-approximation of the domain-bounds of each variable considered as a floating-point number
  - An over-approximation of the error associated with the variable (difference between floating-point and real number values)
  - The contribution of each instruction to the error

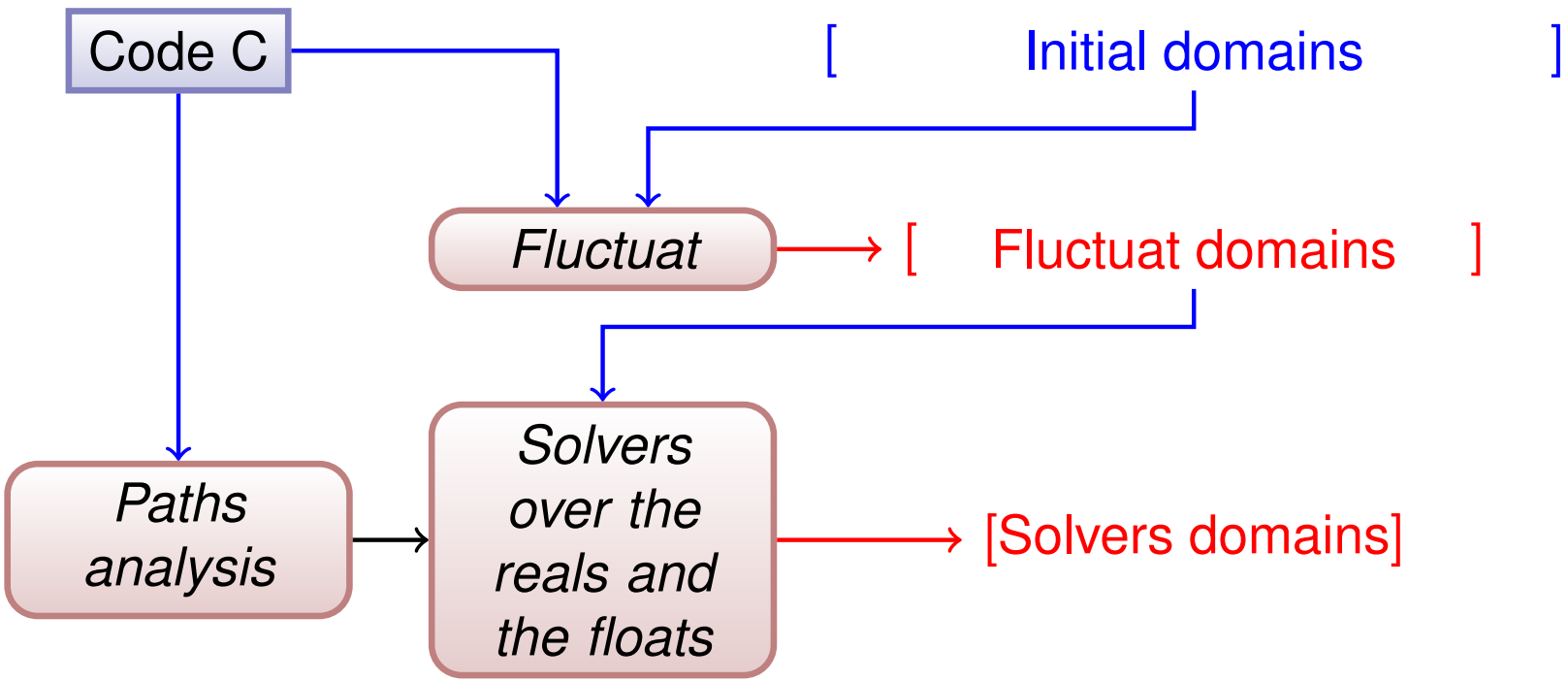
Zonotopes

- **Intuition:** convex polytopes with a central symmetry
  Sets of affine forms $x = x_0 + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n$
  with $\varepsilon_i \in [-1, 1]$
- **Advantages:**
  - Linear correlations between variables are preserved
  - Nonlinear operations are over-approximated by introducing an error term
  - Good trade-off between performance and precision
- **Limits:**
  - Better than the intervals, not as good as polyhedra
  - Not very accurate for nonlinear terms
  - Not accurate on very common program constructions such as if

Proposed approach

Overview

We use constraint-based local consistencies to reduce the domains of variables computed by Fluctuat



Proposed approach

Details

- **Set of CSP** generated for a C program:
  - A CSP is built "on the fly" while exploring a path
    Inconsistent CSP $\rightarrow$ current path is cut off
  - Loops are unfolded a finite number of times

- **Filtering:**
  - **Reals:** Hull & Box consistency – RealPaver
  - **Floats:** 3B consistency – FPCS

- Reduced domain of a variable: union of the intervals generated for this variable while filtering **all successful paths** of the program

Experiments

- Programs
  - `quadratic`: computing the roots of a quadratic equation (GSL library) – conditionals
  - `sinus7`: expression of the 7th-order Taylor series of function sinus – nonlinearity
  - `rump`: polynomial of Rump – nonlinearity
  - `sqrt`: square root computation (Babylonian method) – iterative program
- Expected loss of accuracy of Fluctuat
  - Union at the earliest of program states (join operator): `quadratic`, `sqrt`;
  - Domain intersection due to conditional statements (meet operator): `quadratic`, `sqrt`;
  - Interpolation by expanding approximations due to loops (widening operator): `sqrt`;
  - Nonlinear expression approximation: `quadratic`, `sinus7`, `rump`, `sqrt`.

Results over the reals

	Fluctuat (AI)		RealPaver (CP)	
	Domain	Time	Domain	Time
quadratic$_1$ x_0	$[-\infty, \infty]$	0.1 s	$[-\infty, 0]$	1.5 s
quadratic$_1$ x_1	$[-\infty, \infty]$	0.1 s	$[-8.011, \infty]$	1.5 s
quadratic$_2$ x_0	$[-2e6, 0]$	0.1 s	$[-1e6, 0]$	0.5 s
quadratic$_2$ x_1	$[-1e6, 0]$	0.1 s	$[-5.186e5, 0]$	0.5 s
sinus7	$[-1.009, 1.009]$	0.1 s	$[-0.842, 0.843]$	0.3 s
rump	$[-1e37, 2e37]$	0.1 s	$[-1e36, 1.7e37]$	1.2 s
sqrt$_1$	$[2.116, 2.354]$	0.1 s	$[2.121, 2.346]$	0.3 s
sqrt$_2$	$[2.098, 3.435]$	0.1 s	$[2.232, 3.165]$	0.5 s

FPCS

Correct solver over the floats based on 2B-consistency: no solution lost

- Projection functions for floats:
  - Direct projection: straightforward adaptation of interval arithmetic
  - Inverse projection: uses a larger format than the system variables
- Handling of rounding modes, nonlinear expressions and the usual mathematical functions (trigonometric...)
- Handling of x86 architecture specifics

Results over the floats

	Fluctuat (AI)		FPCS (CP)	
	Domain	Time	Domain	Time
quadratic$_1$ x_0	$[-\infty, \infty]$	0.1 s	$[-\infty, 0]$	0.3 s
quadratic$_1$ x_1	$[-\infty, \infty]$	0.1 s	$[-8.064, \infty]$	0.3 s
quadratic$_2$ x_0	$[-2e6, 0]$	0.1 s	$[-2e6, 0]$	0.3 s
quadratic$_2$ x_1	$[-1e6, 0]$	0.1 s	$[-2503.8, 0]$	0.3 s
sinus7	$[-1.009, 1.009]$	0.1 s	$[-0.853, 0.852]$	0.2 s
rump	$[-1e37, 2e37]$	0.1 s	$[-1e37, 2e37]$	0.2 s
sqrt$_1$	$[2.116, 2.354]$	0.1 s	$[2.120, 2.347]$	1 s
sqrt$_2$	$[-\infty, \infty]$	0.1 s	$[2.232, 3.168]$	1.6 s

Conclusion

- CP
  - Advantages:
    → Good **refutation** capabilities
    → Handling nonlinear constraints
  - Limits: Distinct exploration of each executable path is a **critical issue**
- AI
  - Advantages:
    → Good **scaling** capabilities
    → Zonotopes are better approximations of linear constraints than boxes
  - Limits: **Over-approximations** can be very rough

→ Complementary techniques→ hybrid approach

- Greater domain reduction achieved when combining both approaches
- Automated and tight cooperation is promising

Abstract domain intersection

```
1/* Pre−condition :  x ∈ [0, 10] */
2double conditional(double x) {
3  double y = x*x - x;
4  if (y >= 0)
5    y = x/10;
6  else
7    y = x*x + 2;
8  return y;
9}
```

Quadratic equation roots

```
int quadratic(double a, double b, double c) {
  double r, sgnb, temp, r1, r2
  double disc = b * b - 4 * a * c;
  if (a == 0) {
    if (b == 0)
      return 0;
    else {
      x0 = -c / b;
      return 1;
    }
  }
  if (disc > 0) {
    if (b == 0) {
      r = fabs (0.5 * sqrt (disc) / a);
      x0 = -r;
      x1 =  r;
    } else {
      sgnb = (b > 0 ? 1 : -1);
      temp = -0.5 * (b + sgnb * sqrt (disc));
      r1 = temp / a;
      r2 = c / temp;
      if (r1 < r2) {
        x0 = r1;
        x1 = r2;
      } else {
        x0 = r2;
        x1 = r1;
      }
    }
    return 2;
  } else if (disc == 0) {
    x0 = -0.5 * b / a;
    x1 = -0.5 * b / a;
    return 2;
  } else
    return 0;
}
```

7^{th}-order Taylor series of function sinus

```
double sinus(double x)
{ return x - x*x*x/6 + x*x*x*x*x/120 + x*x*x*x*x*x*x/5040; }
```

Rump's polynomial

```
double rump(double x, double y) {
  double f;
  f = 333.75*y*y*y*y*y*y;
  f = f + x*x*(11*x*x*y*y - y*y*y*y*y*y - 121*y*y*y*y - 2);
  f = f + 5.5*y*y*y*y*y*y*y*y;
  f = f + x / (2*y);
  return f;
}
```

Square root function

```
double sqrt(double x) {
  double xn, xn1;
  xn = x/2;
  xn1 = 0.5*(xn + x/xn);
  while (xn-xn1 > 1e-2) {
    xn = xn1;
    xn1 = 0.5*(xn + x/xn);
  }
  return xn1;
}
```