Constraints over Intervals

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OVERVIEW

◇ **Interval Programming**

- Interval arithmetic
- Interval Analysis methods

◇ **Constraint Programming**

- Overall scheme
- Local consistencies
- Quantified constraints
- Global Constraints

1. Interval Programming

$\longrightarrow$ Basics on interval arithmetics

$\longrightarrow$ Interval Newton-like methods for solving a multi-variate system of non-linear equations

1.1 INTERVAL ARITHMETICS : NOTATIONS

- $\mathbf{c_j}(\mathbf{x_1}, \ldots, \mathbf{x_n})$: a relation over the reals; $\mathcal{C}$: the set of constraints
- $\mathbf{X}$ or $\mathbf{D}_x$: the domain of variable $\mathbf{x}$; $\mathcal{D}$: the set of domains of all the variables
- $I\!R$: the set of real numbers; $I\!F$: the set of floating point numbers
  $\mathbf{a}^+$ (resp. $\mathbf{a}^-$) : the smallest (resp. largest) number of $I\!F$ strictly greater (resp. lower) than $\mathbf{a}$
- $\mathbf{X} = [\underline{\mathbf{X}}, \overline{\mathbf{X}}]$ is the set of real numbers $\mathbf{x}$ verifying $\underline{\mathbf{X}} \leq \mathbf{x} \leq \overline{\mathbf{X}}$
- $\mathbf{x}, \mathbf{y}$: real variables or vectors; $\mathbf{X}, \mathbf{Y}$ interval variables or vectors

1.1 Interval arithmetics : basic definitions (1)

Interval arithmetic (Moore-1966) is based on the representation of variables as intervals

Let f be a real-valued function of n unknowns $\mathbf{X} = \{x_1, \ldots, x_n\}$, an **interval evaluation** $\mathbf{F}$ of f for given ranges $\{X_1, \ldots, X_n\}$ for the unknowns is an interval Y such that

$$\forall \mathbf{X} = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\} \in \mathcal{X} = \{\mathbf{X_1}, \ldots, \mathbf{X_n}\} \quad \underline{\mathbf{Y}} \leq \mathbf{f(X)} \leq \overline{\mathbf{Y}}$$

$\underline{\mathbf{Y}}, \overline{\mathbf{Y}}$ are lower and upper bounds for the values of f when the values of the unknowns are restricted to the box $\mathcal{X}$

1.1 INTERVAL ARITHMETICS : BASIC DEFINITIONS (2)

A relation over the intervals $C : \mathcal{I}^n \rightarrow \mathcal{B}ool$ is an interval extension of the relation $c : \mathcal{R}^n \rightarrow \mathcal{B}ool$ iff:

$$\forall \mathbf{I_1}, \ldots, \mathbf{I_n} \in \mathcal{I} : \mathbf{r_1} \in \mathbf{I_1}, \ldots, \mathbf{r_n} \in \mathbf{I_n} \,\&\, \mathbf{c}(\mathbf{r_1}, \ldots, \mathbf{r_n}) \Rightarrow \mathbf{C}(\mathbf{I_1}, \ldots, \mathbf{I_n})$$

For instance, $\mathbf{I_1} \doteq \mathbf{I_2} \Leftrightarrow (\mathbf{I_1} \cap \mathbf{I_2}) \neq \emptyset$ is an interval extension of the relation $=$ over the real numbers

INTERVAL ARITHMETICS : NATURAL INTERVAL EXTENSION (1)

$\longrightarrow$ In general, it is not possible to compute **the exact enclosure** of the range for an arbitrary function over the real number

$\longrightarrow$ The interval extension of a function is an interval function that computes an **outer approximation** of the range of the function over a domain

INTERVAL ARITHMETICS : NATURAL INTERVAL EXTENSION (2)

The natural interval extension of a real function f is defined by replacing all the mathematical operators in f by their interval equivalents to obtain $\mathbf{F}$

- $[\mathbf{a}, \mathbf{b}] \ominus [\mathbf{c}, \mathbf{d}] = [\mathbf{a} - \mathbf{d}, \mathbf{b} - \mathbf{c}]$
- $[\mathbf{a}, \mathbf{b}] \oplus [\mathbf{c}, \mathbf{d}] = [\mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{d}]$
- $[\mathbf{a}, \mathbf{b}] \otimes [\mathbf{c}, \mathbf{d}] = [\min(\mathbf{ac}, \mathbf{ad}, \mathbf{bc}, \mathbf{bd}), \max(\mathbf{ac}, \mathbf{ad}, \mathbf{bc}, \mathbf{bd})]$
- $[\mathbf{a}, \mathbf{b}] \oslash [\mathbf{c}, \mathbf{d}] = [\min(\frac{\mathbf{a}}{\mathbf{c}}, \frac{\mathbf{a}}{\mathbf{d}}, \frac{\mathbf{b}}{\mathbf{c}}, \frac{\mathbf{b}}{\mathbf{d}}), \max(\frac{\mathbf{a}}{\mathbf{c}}, \frac{\mathbf{a}}{\mathbf{d}}, \frac{\mathbf{b}}{\mathbf{c}}, \frac{\mathbf{b}}{\mathbf{d}})]$ ***if*** $\mathbf{0} \notin [\mathbf{c}, \mathbf{d}]$

Interval extension : properties

$\longrightarrow$ If $0 \notin F(\mathcal{X})$, then no value exists in the box $\mathcal{X}$ such that $f(\mathbf{X}) = 0$
$\Leftrightarrow$ **the equation $f(\mathrm{X})$ has no root in the box $\mathcal{X}$**

$\longrightarrow$ Interval arithmetics can be implemented taking into account round-off errors

$\longrightarrow$ Interval arithmetic preserves **inclusion monotonicity**:
$\mathbf{Y} \subseteq \mathbf{X} \quad \Rightarrow \quad \mathbf{F}(\mathbf{Y}) \subseteq \mathbf{F}(\mathbf{X})$ but interval arithmetics is **sub-distributive**:
$\mathbf{X}(\mathbf{Y} + \mathbf{X}) \subseteq \mathbf{XY} + \mathbf{XZ}$

INTERVAL EXTENSION : LIMITATIONS

→ The **wrapping effect**, which overstimates by a unique vector the image of an interval vector (which is in general not a vector)

→ The **dependency problem**, which is due to the independence the different occurences of some variable during the interval evaluation of an expressionExample :
Consider $X = [0, 5]$
$\mathbf{X - X = [0 - 5, 5 - 0] = [-5, 5]}$ instead of $[0, 0]$!$\mathbf{X^2 - X = [0, 25]}$ -

INTERVAL EXTENSION : USING DIFFERENT LITERAL FORMS (1)

$\longrightarrow$ Factorized form (Horner for polynomial system) or distributed form

$\longrightarrow$ First-order Taylor development of $\mathbf{f}$

$$\mathbf{F}_{\text{tay}}(\mathbf{X}) = \mathbf{f}(\mathbf{x}) + \mathbf{J}(\mathbf{X}).(\mathbf{X} - \mathbf{x})$$

with $\forall \mathbf{x} \in \mathbf{X}$, $\mathbf{J}()$ being the Jacobian of $\mathbf{f}$

INTERVAL EXTENSION : USING DIFFERENT LITERAL FORMS (2)

$\longrightarrow$ In general, first order Taylor extensions yield a better enclosure than the natural extension on small intervals

$\longrightarrow$ Taylor extensions have a quadratic convergence whereas the natural extension has a linear convergence

$\longrightarrow$ In general, neither $\mathbf{F}_{\text{nat}}$ nor $\mathbf{F}_{\text{tay}}$ won't allow to compute the exact range of a function $\mathbf{f}$

INTERVAL EXTENSION : USING DIFFERENT LITERAL FORMS (3)

Consider $f(x) = 1 - x + x^2$, and $\mathbf{X} = [0, 2]$

$$\mathbf{f}_{\text{tay}}([\mathbf{0}, \mathbf{2}]) = \mathbf{f}(\mathbf{x}) + (\mathbf{2X} - \mathbf{1})(\mathbf{X} - \mathbf{x})$$
$$= \mathbf{f}(\mathbf{1}) + (\mathbf{2}[\mathbf{0}, \mathbf{2}] - \mathbf{1})([\mathbf{0}, \mathbf{2}] - \mathbf{1}) = [-\mathbf{2}, \mathbf{4}]$$

$$\mathbf{f}([\mathbf{0}, \mathbf{2}]) = \mathbf{1} - \mathbf{X} + \mathbf{X}^{\mathbf{2}} = \mathbf{1} - [\mathbf{0}, \mathbf{2}] + [\mathbf{0}, \mathbf{2}]^{\mathbf{2}} = [-\mathbf{1}, \mathbf{5}]$$

$$\mathbf{f}_{\text{factor}}([\mathbf{0}, \mathbf{2}]) = \mathbf{1} + \mathbf{X}(\mathbf{X} - \mathbf{1}) = \mathbf{1} + [\mathbf{0}, \mathbf{2}]([\mathbf{0}, \mathbf{2}] - \mathbf{1}) = [-\mathbf{1}, \mathbf{3}]$$

whereas the range of f over $\mathbf{X} = [\mathbf{0}, \mathbf{2}]$ is $[\mathbf{3}/\mathbf{4}, \mathbf{3}]$

1.2 Interval Analysis methods

Goal : to determine the zeros of a system of n equations $f_i(x_1, \ldots, x_n)$ in n unknowns x_i inside the interval vector $X = \{X_1, \ldots, X_n\}$ with $x_i \in X_i$ for $i = 1, \ldots, n$

$\longrightarrow$ **Gauss-Seidel iterative method**

$\longrightarrow$ **Interval Newton algorithm**

Gauss-Seidel iterative method

Consider the case of interval linear equations :

$$\mathbf{A.x = b}$$

with $\mathbf{A}$ an interval matrix and $\mathbf{b}$ an interval vector

For each unknowns X_i, the Gauss-Seidel algorithm is defined by the following iterative process:

$$\mathbf{X_i^{k+1} = (b_i - \sum_{j=1}^{i-1} A_{i,j} X_j^{k+1} - \sum_{j=i+1}^{n} A_{i,j} X_j^{k}) / A_{i,i} \cap X_i^k}$$

Pre-conditioning $\rightarrow$ to shrink the width of the intervals

INTERVAL NEWTON ALGORITHM (1)

Principle of the Newton operator :

Consider $\mathbf{f} : \mathcal{R} \rightarrow \mathcal{R}$, the mean value theorem says :

$\exists \mathbf{a} \in [\mathbf{v}, \mathbf{u}] : \mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v}) = (\mathbf{u} - \mathbf{v})\mathbf{f}'(\mathbf{a})$ and thus,

$\mathbf{f} \ : \ \mathbf{v} = \mathbf{u} - \frac{\mathbf{f}(\mathbf{u})}{\mathbf{f}'(\mathbf{a})}$ if $\mathbf{v}$ is a zero of $\mathbf{f}$

If $\mathbf{a} \in \mathbf{I}$ then $\mathbf{f}(\mathbf{a}) \in \mathbf{F}(\mathbf{I})$, and $\mathbf{v} \in \tilde{\mathbf{u}} - \frac{\mathbf{F}(\tilde{\mathbf{u}})}{\mathbf{F}'(\mathbf{I})} = \mathbf{N}(\mathbf{F}, \mathbf{F}', \tilde{\mathbf{u}}, \mathbf{I})$

If $\mathbf{v}$ is a zero of $\mathbf{f}$ then $\mathbf{v} \in \mathbf{I_n} \quad (\mathbf{n} \geq \mathbf{1})$ where

$\mathbf{I_0} = \mathbf{I}$

$\mathbf{I_{i+1}} = \mathbf{N}(\mathbf{F}, \mathbf{F}', \mathbf{center}(\mathbf{I_i}), \mathbf{I}) \cap \mathbf{I_i}$

$\mathbf{I_n} = \mathbf{I_{n+1}}$

INTERVAL NEWTON ALGORITHM (2)

The Interval Newton algorithm is used to solve **non-linear systems** with

$$\mathbf{X_{k+1}} = \mathbf{N}(\tilde{\mathbf{x}}_\mathbf{k}, \mathbf{X_k}) \cap \mathbf{X_k} \text{ with } \mathbf{N}(\tilde{\mathbf{x}}_\mathbf{k}, \mathbf{X_k}) = \tilde{\mathbf{x}}_\mathbf{k} - \mathbf{A}.\mathbf{f}(\tilde{\mathbf{x}}_\mathbf{k})$$

where $\mathbf{A} = [\mathbf{F}^{'}(\mathbf{X_k})]^{-1}$ and $\tilde{\mathbf{x}}_\mathbf{k} \in \mathbf{X_k}$ (e.g., the mid-point of $\mathbf{X_k}$)

Properties :

◇ If $\mathbf{N}(\tilde{\mathbf{x}}_\mathbf{k}, \mathbf{X_k}) \cap \mathbf{X_k} = \emptyset$, then the system $\mathbf{F}$ has no solution in $\mathbf{X_k}$

◇ if $\mathbf{N}(\tilde{\mathbf{x}}_\mathbf{k}, \mathbf{X_k})_\mathbf{k} \subset \mathbf{X_k}$, there is one or more solution in X_{k+1}

Matrix $\mathbf{A} = [\mathbf{F}^{'}(\mathbf{X_k})]^{-1}$ may be costly to compute

... to determine $N(\tilde{x}_k, X_k) \rightarrow$ solve the linear system:

$$\mathbf{F}^{'}(\mathbf{X_k})(\mathbf{N}(\tilde{\mathbf{x}}_\mathbf{k}, \mathbf{X_k}) - \tilde{\mathbf{x}}_\mathbf{k}) = -\mathbf{f}(\tilde{\mathbf{x}}_\mathbf{k})$$

2. Constraint Programming

Numeric CSP $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ **:**

◇ $\mathcal{X} = \{x_1, \ldots, x_n\}$ is a set of variables

◇ $\mathcal{D} = \{D_{x_1}, \ldots, D_{x_n}\}$ is a set of domains
(D_{x_i} contains all acceptable values for variable x_i)

◇ $\mathcal{C} = \{c_1, \ldots, c_m\}$ is a set of constraints

2.1 OVERALL SCHEME

The constraint programming framework is based on a **branch & prune** schema which is best viewed as an iteration of two steps:

1. **Pruning the search space**

2. **Making a choice to generate two (or more) sub-problems**

◇ The pruning step $\longrightarrow$ **reduces an interval** when it can prove that the upper bound or the lower bound does not satisfy some constraint

◇ The branching step $\longrightarrow$ **splits the interval** associated to some variable in two intervals (often with the same width)

2.2 LOCAL CONSISTENCIES (1)

→ Informally speaking, a constraint system C satisfies a partial consistency property if **a relaxation of C is consistent**

Consider $\mathbf{X} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ and $\mathbf{C}(\mathbf{x}, \mathbf{x_1}, \ldots, \mathbf{x_n}) \in \mathcal{C}$: if $\mathbf{C}(\mathbf{x}, \mathbf{x_1}, \ldots, \mathbf{x_n})$ does not hold for any values $\mathbf{a} \in [\underline{\mathbf{x}}, \mathbf{x}']$, then X may be shrinked to $\mathbf{X} = [\mathbf{x}', \overline{\mathbf{x}}]$

→ A **constraint C_j is AC-like-consistent** if for any variable x_i in $\mathcal{X}_j$, **the bounds $\underline{D_i}$ and $\overline{D_i}$ have a support** in the domains of all other variables of $\mathcal{X}_j$

Local consistencies used in `BNR-prolog`, `Interlog`, `CLP(BNR)`, `PrologIV`, `UniCalc`, `Ilog Solver`, `Numerica`, `Icos`, `RealPaver` are AC-like-consistencies

2.2 LOCAL CONSISTENCIES (2)

Local consistencies are conditions that filtering algorithms must satisfy

$\longrightarrow$ fixed point algorithm defined by the sequence $\{\mathcal{D}_k\}$ of domains generated by the iterative application of an operator

$$\mathbf{Op} : \mathbb{I}(\mathbb{R})^{\mathbf{n}} \longrightarrow \mathbb{I}(\mathbb{R})^{\mathbf{n}}$$

$$\boxed{\mathcal{D}_k = \begin{cases} \mathcal{D} & \text{if } k = 0 \\ Op(\mathcal{D}_{k-1}) & \text{if } k > 0 \end{cases}}$$

2.2 LOCAL CONSISTENCIES (3)

Properties of the operator Op :

- $\diamond$ $\mathbf{Op}(\mathcal{D}) \subseteq \mathcal{D}$ (contractance)
- $\diamond$ $\mathbf{O}p$ is conservative; that is, it cannot remove any solution
- $\diamond$ $\mathcal{D}' \subseteq \mathcal{D} \Rightarrow \mathbf{Op}(\mathcal{D}') \subseteq \mathbf{Op}(\mathcal{D})$ (monotonicity)

The limit of the sequence $\{\mathcal{D}_k\}$, which corresponds to the greatest fixed point of the operator $\mathbf{O}p$

2.2 LOCAL CONSISTENCIES (4)

→ **2B–consistency** (also known as hull consistency) only requires to check the Arc–Consistency property **for each bound** of the intervals

→ **Box–consistency** is a coarser relaxation of Arc–Consistency than 2B–consistency ... **but Box–consistency algorithms actually achieve a stronger filtering than 2B–consistency**

→ **Strong consistencies** : no bound of the domains can be removed with a local consistency filtering algorithm

2B–CONSISTENCY (1)

Variable x is 2B–consistency for constraint $\mathbf{f(x, x_1, \ldots, x_n) = 0}$ if the lower (resp. upper) bound of the domain $\mathbf{X}$ is the smallest (resp. largest) solution of $\mathbf{f(x, x_1, \ldots, x_n)}$

Definition : 2B–consistency

Let $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ be a CSP and $\mathbf{C} \in \mathcal{C}$ a k-ary constraint over $(\mathbf{X_1, \ldots, X_k})$

$\mathbf{C}$ is 2B–consistency iff :

$\forall \mathbf{i}, \mathbf{X_i} = \Box\{\mathbf{\tilde{x}_i} \mid \exists \mathbf{\tilde{x}_1} \in \mathbf{X_1}, \ldots, \exists \mathbf{\tilde{x}_{i-1}} \in \mathbf{X_{i-1}}, \quad \exists \mathbf{\tilde{x}_{i+1}} \in \mathbf{X_{i+1}}, .$

such that

$\mathbf{c(\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_i, \tilde{x}_{i+1} \ldots, \tilde{x}_k)}$ holds$\}$

A CSP is 2B–consistency iff all its constraints are 2B–consistency

BOX–CONSISTENCY (1)

Variable $\mathbf{x}$ is Box–Consistent for constraint $\mathbf{f(x, x_1, \ldots, x_n) = 0}$ if the bounds of the domain of x correspond to the leftmost and the rightmost zero of the optimal interval extension of $\mathbf{f(x, x_1, \ldots, x_n)}$

Definition : Box–consistency

Let $(\mathcal{X}, \mathcal{D}, \mathcal{C})$be a CSP and $\mathbf{C} \in \mathcal{C}$ a k-ary constraint over $\mathbf{(X_1, \ldots, X_k)}$

$\mathbf{C}$ is Box–Consistent if, for all $\mathbf{X_i}$ the following relations hold :

1. $\mathbf{C(X_1, \ldots, X_{i-1}, [\underline{X_i}, \underline{X_i}^+), X_{i+1}, \ldots, X_k)}$
2. $\mathbf{C(X_1, \ldots, X_{i-1}, (\overline{X_i}^-, \overline{X_i}], X_{i+1}, \ldots, X_k)}$

LOCAL CONSISTENCY FILTERING (1)

Algorithms that achieve a local consistency filtering are based upon projection functions

- **Solution functions** expresse the variable $\mathbf{x_i}$ in terms of the other variables of the constraint. The solution functions of $\mathbf{x + y = z}$ are: $\mathbf{f_x = z - y, f_y = z - x, f_z = x + y}$

- An approximation of the projection of the constraint over $\mathbf{X_i}$ given a domain $\mathcal{D}$ can be computed with any interval extension of this solution function $\rightarrow$ we have a way to compute $\pi_{\mathbf{j,i}}(\mathcal{D})$

- For complex constraints, no analytic solution function may exist Consider $\mathbf{x + \log(x) = 0}$

LOCAL CONSISTENCY FILTERING (2)

1. Analytic functions always exist when the variable to express in terms of the others appears only once in the constraint $\longrightarrow$ **considers that each occurrence is a different new variable**

   For $\mathbf{x} + \mathbf{log}(\mathbf{x}) = \mathbf{0}$ we obtain $\mathbf{x_1} + \log(\mathbf{x_2}) = \mathbf{0}$
   Thus $\mathbf{f_{x_1}} = -\log(\mathbf{x_2})$, $\mathbf{f_{x_2}} = \exp^{-\mathbf{x_1}}$
   and $\pi_{\mathbf{x}+\mathbf{log}(\mathbf{x})=\mathbf{0},\mathbf{x}}(\mathbf{X}) = -\log(\mathbf{X}) \cap \exp^{-\mathbf{X}}$

   - ◇ This approach is used for computing **2B–consistency filtering** (the initial constraints are decomposed into primitive constraints)
   - ◇ Decomposition does not change the semantics of the initial constraints system but **it amplifies the dependency problem**

LOCAL CONSISTENCY FILTERING (3)

2. Transformation of the constraint $\mathbf{C_j(x_{j_1}, ...x_{j_k})}$ into $\mathbf{k}$ mono-variable constraints $\mathbf{C_{j,l}, l = 1 \ldots k}$ by substituting their intervals for the other variables

$\longrightarrow$ The two extremal zeros of $\mathbf{C_{j,l}}$ can be found by a **dichotomy algorithm combined with a mono-variable version of the interval Newton method**

$\longrightarrow$ This approach is **well adapted for Box–consistency filtering**

LOCAL CONSISTENCY FILTERING (4)

3. Use the Taylor extension to transform the constraint into an interval linear constraint. $f(X) = 0$ becomes

$$\mathbf{f(c) + \sum_{i=1}^{n} nat(\frac{\partial f}{\partial x_i})(X) * (X_i - c_i) = 0}$$

where $\mathbf{c = m(X)}$. The derivatives are evaluated over a box $\mathcal{D}$ that contains $\mathbf{X}$, $\mathcal{D}$ is considered as constant, and with $\mathbf{c = m(\mathcal{D})}$

$\longrightarrow$ The equation becomes an interval linear equation in X, which does not contain multiple occurrences

$\longrightarrow$ **Solving the squared interval linear system allows much more precise approximations of projections to be computed**

STRONGER CONSISTENCIES, 3B–CONSISTENCY (1)

3B–Consistency, a relaxation of path consistency, checks whether 2B–Consistency can be enforced when the domain of a variable is reduced to the value of one of its bounds in the whole system

STRONGER CONSISTENCIES, 3B–CONSISTENCY (2)

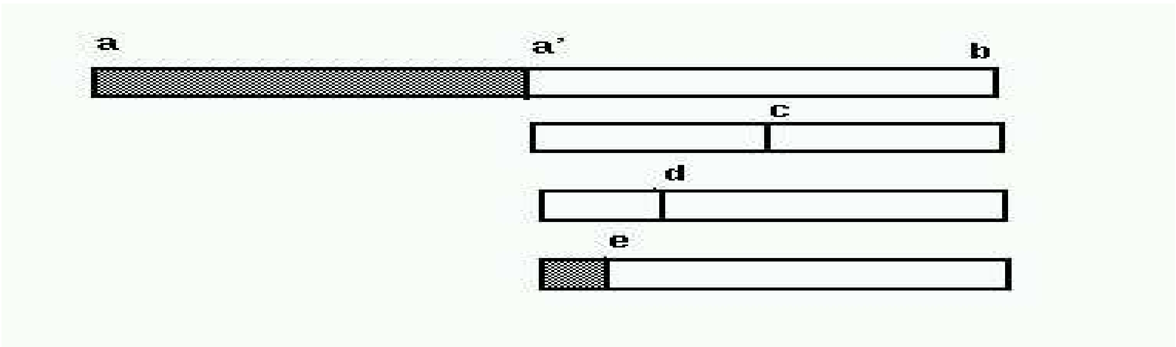
Definition : 3B–Consistency

Let $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ be a CSP and $\mathbf{x}$ a variable of $\mathcal{X}$ with $\mathbf{D_x} = [\mathbf{a}, \mathbf{b}]$. Let also:

- ◇ Let $\mathbf{P}_{\mathbf{D^1_x} \leftarrow [\mathbf{a}, \mathbf{a}^+)}$ be the CSP derived from $\mathbf{P}$ by substituting $\mathbf{D_x}$ in $\mathcal{D}$ with $\mathbf{D^1_x} = [\mathbf{a}, \mathbf{a}^+)$
- ◇ Let $\mathbf{P}_{\mathbf{D^2_x} \leftarrow (\mathbf{b}^-, \mathbf{b}]}$ lbe the CSP derived from $\mathbf{P}$ by substituting $\mathbf{D_x}$ in $\mathcal{D}$ with $\mathbf{D^2_x} = (\mathbf{b}^-, \mathbf{b}]$

$\mathbf{X}$ is 3B–Consistent iff $\mathbf{\Phi_{2B}}(\mathbf{P}_{\mathbf{x} \leftarrow [\underline{\mathbf{x}}, \underline{\mathbf{x}}^+)}) \neq \mathbf{P}_\emptyset$ and $\mathbf{\Phi_{2B}}(\mathbf{P}_{\mathbf{x} \leftarrow (\overline{\mathbf{x}}^-, \overline{\mathbf{x}}]}) \neq \mathbf{P}_\emptyset$

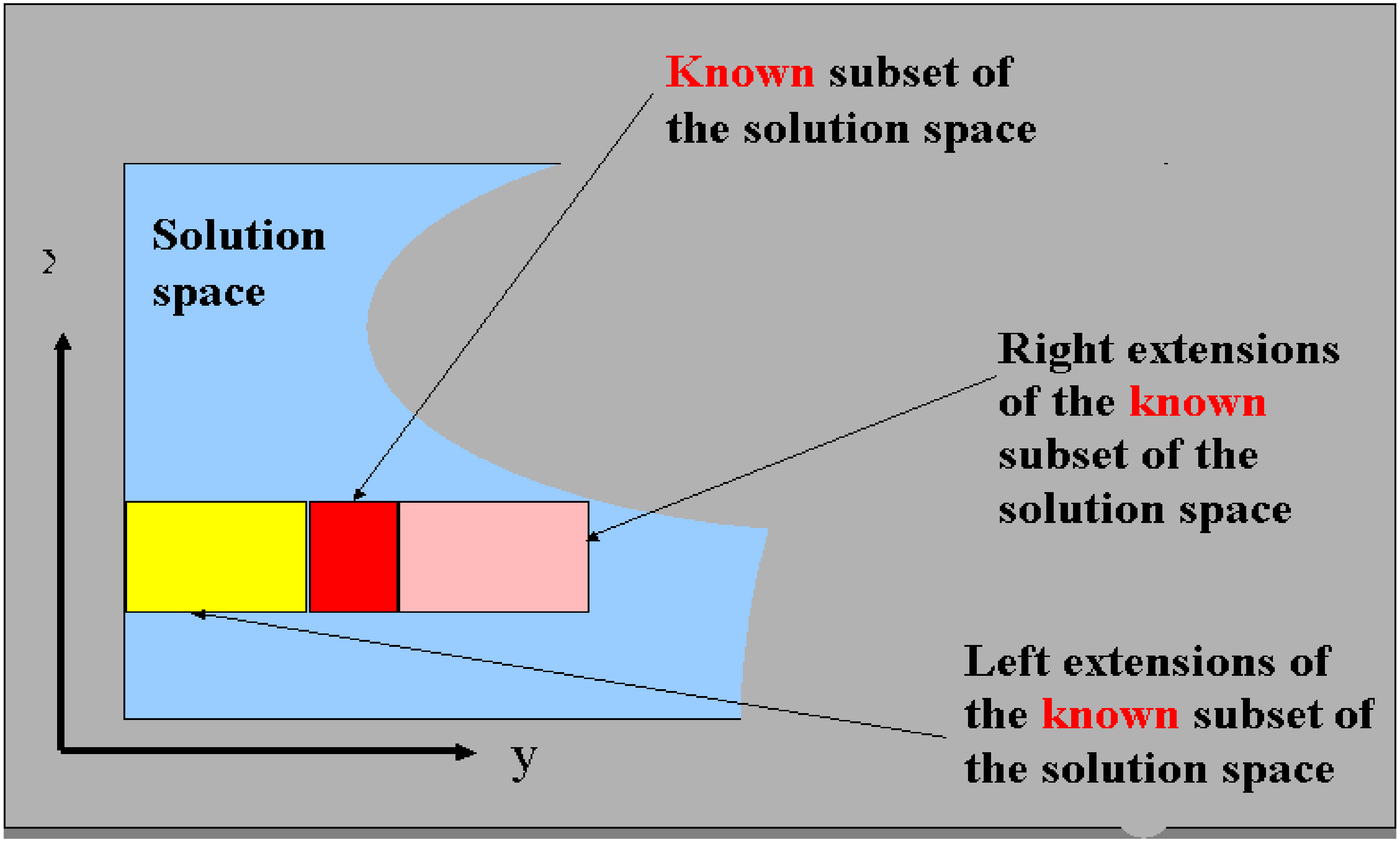
STRONGER CONSISTENCIES, 3B–CONSISTENCY (3)



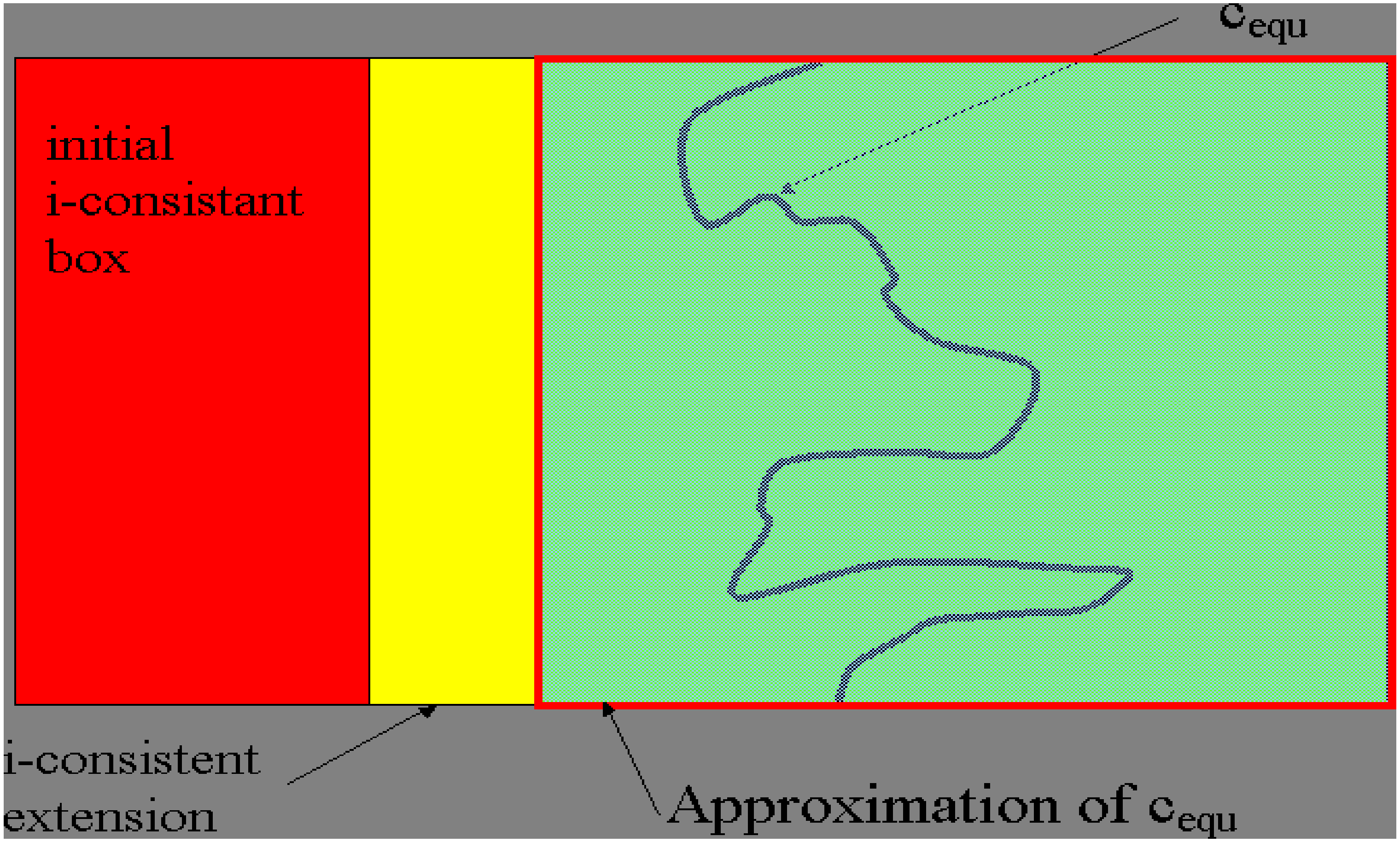
Let $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ be a CSP and $\mathbf{D_x = [a, b]}$, if $\mathbf{\Phi_{2B}(P_{D_x \leftarrow [a, \frac{a+b}{2}]}) = \emptyset}$

- ◇ then the part $\mathbf{[a, \frac{a+b}{2})}$ de $\mathbf{D_x}$ will be removed and the filtering process continues on the interval $\mathbf{[\frac{a+b}{2}, b]}$
- ◇ otherwise, the filtering process continues on the interval $\mathbf{[a, \frac{3a+b}{4}]}$.

QUANTIFIED CONSTRAINTS (1)



QUANTIFIED CONSTRAINTS (2)



QUANTIFIED CONSTRAINTS (3)

◇ $\forall x \in D_x : \; x + x_1 \; < 5 \quad \text{with} \quad D_x = [-2, 2], D_{x_1} \; = [1, 5]$

$\neg(\forall x \in D_x : \; x + x_1 \; < 5) - > x + [1, 5] \geq 5 \; \Rightarrow x \geq 0$

$D_x \in [-2, 0) \Rightarrow \forall x \in D_x : \; x + x_1 \; < 5$ holds ▮

◇ $\forall (x \in X) \quad \forall (y \in Y) \quad \exists (z \in Z) : \quad (z = x + y)$ ▮

◇ Modal intervals

Global Constraints (1)

◇ **"Syntactical" approach**

To handle an approximation of the whole constraint system with the **simplex algorithm**

⟶ replace each non linear term by a new variable ▮

⟶ introduce redundant linear constraints to get a **tight approximation** of the non-linear terms ▮

⟶ solving a linear relaxation with the simplex algorithm

GLOBAL CONSTRAINTS (2)

◇ **“Semantic” approach**

⟶ Distance constraint