A Fixed Point Computation of Partial Correlation Coefficients

J. LE ROUX and C. GUEGUEN

IEEE Trans. on ASSP, vol. 27, no 3, June 1977, pp 257-259

Abstract - **This paper introduces a new computational algorithm for the partial correlation coefficients of a linear system given the covariance of its output when excited by a white input noise. Although derived from Levinson's well-known procedure, the proposed algorithm does not make use of the usual parameters in the linear prediction recursion. It may be implemented using fixed point arithmetics. Application to speech waves is emphasized.** [1]

I. Introduction

Speech analysis makes an extensive use of the so-called linear predictive coding (LPC). This approach can be interpreted as modeling the speech samples s_n as the output of a p-order linear system represented by the autoregressive (AR) equation

$$\sum_{i=0}^{p} a_i s_{n-i} = u_n, \qquad a_0 = 1 \tag{1}$$

when excited by a white Gaussian input noise u_n.

A large number of criteria have been introduced by various authors to match this model to the real time series. These approaches turn out to be equvalent in the linear AR Gaussian case and lead to the well-known Yule-Walker equations [1]-[3].

The main feature about this linear vector equation is the fundamental role played by the autocorrelation matrix R_p, constructed in the Toeptlitz form from the autocorrelation coefficients r_i:

$$r_i = E(s_n s_{n+i}), \qquad r_i = r_{-i} \tag{2}$$

where $E(.)$ denotes the mathematical expectation.

Taking into account this special form of R_p, Levinson [4], Robinson [5], and Durbin [6] have introduced various efficient algorithms to solve this type of equation, instead of using the standard Gauss or Cholesky procedures. These recursions, on the order $h = 1, \dots, p$, of the model, make some useful parameters appear: the partial correlation (PARCOR) coefficients k_h.

These coefficients have provent so be of significant importance (far beyond the algorithmic point of view) in the forward-backward prediction interpretation. They have been extensively used in digital speech transmission and synthesis. But, up to now, the computation of the k_h involved the calculation of the a_i for various orders $h = 1, \dots, p$.

Very little is known about the range of these intermediate variables and this causes bothersome scaling problems.

This paper introduces a new method for computing the PARCOR coefficients through the use of new intermediate variables. The physical meaning of these variables as impulse response sequences or cross correlations solves the scaling problem and gives rise to a fixed point arithmetic implementation.

II. The New Formulation

The inner product formulation, as in [7], will be used to introduce the new formulation. For a comprehensive and tutorial introduction to standard LPC, the reader is referred to [7] and [8].

Denote by $R(z)$ the z-transform of the autocorrelation sequence of the signal and by $A_h(z)$ the polynomial denominator (in z^{-1}) of the transfer function of the AR model

$$R(z) = \sum_{i=-\infty}^{+\infty} r_i z^{-i}, \qquad A_h(z) = \sum_{i=0}^{h} a_i^h z^{-i}. \tag{3}$$

It is ealily shown that the mean-square prediction error of the signal s_n, produced by this hth

[1] Manuscript received February 3, 1976; revised November 30, 1976. The authors are with the Laboratoire de Theorie des Systemes, Ecole Nationale Superieure des Telecommunications, 75634 CEDEX 13 Paris, France.

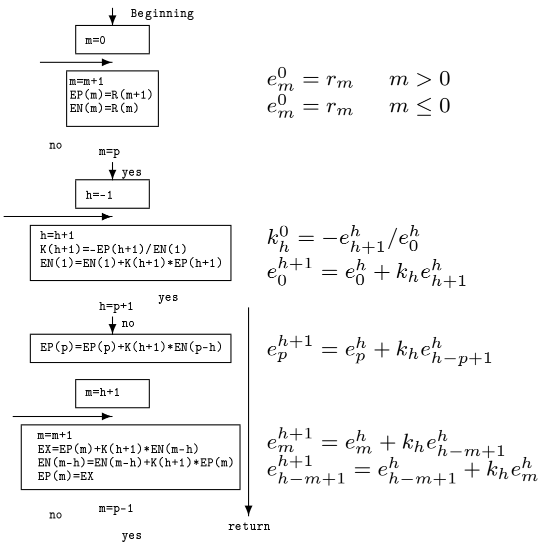

Figure 1: Computation of the PARCOR coefficients $K(1),\ldots,K(p)$ and the error energy EN(1) of a p-order AR model from the correlation sequence $R(0),\ldots,R(p+1)$. For each stage h, the values of e^h_m are stored in EP when $m \geq 0$ and in EN when $m < 0$

order model, is the norm

$$(A_h(z), A_h(z)) = \frac{1}{2\pi j}\int_\Gamma A_h(z)R(z)A_h(z^{-1})z^{-1}dz. \tag{4}$$

The minimization of (4) is obtained using the projection thorem in Hilbert spaces and it gives, as optimality conditions, the Yule-Walker equations in terms of inner products:

$$(A_h(z), z^{-i}) = 0, \qquad i = 1, h. \tag{5}$$

According to Durbin, an efficient way for solving these equations is the following set of recursions:

$$k_h = -\left(\sum_{i=0}^{h} a^h_i r_{h+1-i}\right)/\alpha_h \qquad (a)(6)$$
$$\alpha_{h+1} = \alpha_h(1-k_h^2) \qquad (b)$$
$$a^{h+1}_i = a^h_i + k_h a^h_{h+1-i},\ i = 1, h+1. \quad (c)$$

We now introduce the new intermediate variables by the inner product

$$e^h_i = (A_h(z), z^{-i}) = \sum_{m=0}^{h} a^h_m r_{i-m} \qquad \forall i. \tag{7}$$

According to (1), the result of filtering the data samples s_n by the optimal inverse filter $A_h(z)$ of order h is an estimate $\hat{u}^h_n$ of the input

$$\hat{u}^h_n = \sum_{m=0}^{h} a^h_m s_{n-m}, \qquad a^h_0 = 1. \tag{8}$$

It follows that the parameters (7) may be interpreted as cross-correlations

$$e^h_i = E(\hat{u}^h_n . s_{n-i}). \tag{9}$$

Due to the optimality conditions (5), the e^h_i are such that

$$e^h_i = 0, \qquad i = 1, h. \tag{10}$$

Moreover, these parameters, computed for $i < 0$ from the higher order correlation coefficiens, are estimates of the impulse response of the model. They turn out to coincide with this impulse response if the time series have actually been generted by a system of order h.

It will be shown that these extended Yule-Walker equations [compare (7) and (5)] provide a new recursive algorithm for solving the LPC problem.

III. Recursive Algorithm

Introducting, as in [7], the polynomial $B_h(z)$ such that

$$B_h(z) = z^{-h-1}A_h(z^{-1}),$$

it follows that

$$(B_h(z), z^{-i}) = (z^{i-h-1}, A_h(z)) = e^h_{h+1-i}. \tag{11}$$

As a consequence, assuming $A_h(z)$ to be the solution of (5), the equalities (12) hold for $i = 1, h$:

$$(A_h(z), z^{-i}) = e^h_i = 0, \quad i = 1, h$$
$$(B_h(z), z^{-i}) = e^h_{h+1-i} = 0, \quad i = 1, h. \tag{12}$$

As in the standard solution, the optimal $A_{h+1}(z)$ is constructed as the linear combination with coefficient k_h such that

$$e^{h+1}_i = (A_{h+1}(z), z^{-i}) = 0, \qquad i = 1, h+1 \tag{13}$$
$$A_{h+1}(z) = A_h(z) + k_h B_h(z). \tag{14}$$

Computing the inner product $(A_h(z), z^{-i})$ defined in (7), and using (11) and (12), the recursive solution is expressed in terms of the new variables:

$$k_h = -e^h_{h+1}/e^h_0 \qquad (a)(15)$$
$$e^{h+1}_0 = e^h_0(1-k_h^2) \qquad (b)$$
$$e^{h+1}_i = e^h_i + k_h e^h_{h+1-i} \quad \forall i. \qquad (c)$$

In this solution, (15a) and (15b) do not differ from the previous ones [i.e. (6a) and (6b)] and compute the same essential parameters k_h. But

(15c) now involves the physical variables e_i^h instead of the a_i^h. It will be shown in section IV that by using a straightforward bound on the e_i^h, the algorithm can be automatically scaled and implementdetd according to Fig. 1.

One of the main features of this algorithm is to mix the AR and MA approaches. On the one hand, the usual PARCOR coefficients of the AR model are computed. But on the other hand, part of the computation is done on the e_i^h, which are the MA counterpart of the AR model. In that sense, the proposed algorithm is more a factorization at the level of the impulse response thant an AR modeling technique.

IV. A Fixed Point Implementation

In many applications, the computational time is severely limited and the LPC has to be implemented on a small size digital computer (without any floating point processor). A fixed point implementation is consequently of particular interest.

Using the properties of the inner product formulation [7], the Cauchy-Schwarz inequality applied to (7) gives

$$\begin{aligned} \left|e_i^h\right|^2 &= (A_h(z), z^{-i})^2 \qquad (16) \\ &\leq (A_h(z), A_h(z)).(z^{-i}, z^{-i}) \end{aligned}$$

where

$$(A_h(z), A_h(z)) = e_0^h \quad \text{and} \quad (z^{-i}, z^{-i}) = r_0; \qquad (17)$$

therefore,

$$\left|e_i^h\right|^2 \leq e_0^h.r_0 \qquad \forall i. \qquad (18)$$

Moreover, it has been shown [7] that for exact correlation data,

$$e_i^h = r_0 \prod_{i=0}^{h-1}(1-k_i^2) \text{ and } -1 \leq k_i \leq +1. \qquad (19)$$

Consequently, the following bound holds

$$\left|e_i^h\right| \leq r_0. \qquad (20)$$

Using the addition assumption, $r_0 < 1$, the bound (20) shows that all the intermediate variables lie between -1 and +1. As a consequence, the implementation may be conducted using fixed point arithmetics only. The bound can be extended for higher order $h' > h$. Due to the inequality,

$$e_0^{h'} \leq e_0^h \qquad \forall h' \geq h, \qquad (21)$$

a range of variations for the further e_i^h is

$$\left|e_i^h\right|^2 \leq e_0^h.r_0 \qquad \forall h' \geq h. \qquad (22)$$

Since (15c) is homogeneous in these variables, it is then possible to shift the stored values to the left in order to improve the precision in the following computations.

If between the stages $(h-1)$ and (h) the number of most significant bits to be zero is increased by $2b$, a shift of b bits to the left can then be made on the e_i^h.

V. Experimental Results

This method requires $(p^2 - p + 1)$ multiplications and divisions (i.e. 91 for a 10th-order model) and $2p + 3$ memory cells. It has been implemented using fixed point arithmetics on a standard 16-digits computer (PACER 100 EAI, multiply 5.6 μs). In that case, the calculation of the PARCOR coefficients for a 10-th model requires 3.5 ms. Even if the correlation computation time is much more for speech signal, this allows the LPC to be applied several times for a better estmation (ARMA models, for instance [9]). The left shifts do not improve the results very much (i. e. no significant change on the poles location). The maximum shift reached only two digits. The inequality (20) was always satisfied, and the stability was guaranteed for more than 3000 computations of PARCOR coefficients on real speech waves.

The method has been extensively compared to the usual algorithms implemened using a floating point processor. In more than 100 experiments, the differences between the results is less than 0.005 on k_{10}. The frequency deviation do not exceed 1 Hz for the most significant poles at a sampling rate of 10 Kz. The difference are below the possible deviations occurring in the computations of the autocorrelation coefficients on a vocal signal.

VI. Conclusions

The PARCOR coefficients have proven to be of major interest in speech analysis, synthesis, transmission, and even as intermediate parameters. This approach has also been applied successfully on various types of other signals [10].

This paper has proposed a new computational algorithm for these coefficients by introducing new intermediate variables. The computations are then made on estimated impulse responses with no reference to the AR model coefficients. As a consequence, the proposed algorithm can be implemented using fixed point arithmetics only, and it may contribute to the development of specialized processors achieving the linear prediction in real time for fast signals.

VI. References

[1] B. Atal and S. Hanauer, "Speech analysis and synthesis by linear prediction of the speech wave," *J. Acoust. Soc. Amer.,* vol. 41, pp. 293-309, 1967;

[2] F. Itakura and S. Saito, "Analysis synthesis telephony based on the maximum likelihood method" presented at the 6th Int. Congr. on Acoust., Tokyo, Japan, 1968.

[3] J. D. Markel, "Digital inverse filtering: A new tool for formant trajectories estimation," *IEEE Trans. Audio Electroacoust.,* vol. AU-20, pp. 129-137, Apr. 1972.

[4] N. Levinson, "The Wiener RMS error criterion in filter design and prediction", *J. Math. Phys.* vol. 25, no. 4, p. 261, 1947.

[5] E. A. Robinson, *Statistical Communication and Detection With Special References to Digital Data Processing of Radar and Seismic Signals.* New York: Hafner, 1967, p. 274.

[6] J. Durbin, "The fitting of time-series models, " *Rev. Inst. Int. Stat.",* vol. 28, no. 3, pp. 344-348, 1973.

[7] J. D. Markel and A. H. Gray, " On autocorrelation equations as applied to speech analysis," IEEE Trans. Audio. Electroacoust. vol. AU-21, Apr. 1973.

[8] J. Makhoul, "Linear prediction, A tutorial review," *Proc. IEEE,* vol. 63, Apr. 1975;

[9] C. Gueguen and M. Mathieu, "Contribution des zéros à la modélisation du signal de parole," presented at the 7èmes Journées du Galf, Nancy, France, 1976;

[10] M. Mathieu, "Analyse de l'électro-encéphalogramme par prédiction linéaire," Thèse de Docteur-Ingénieur, ENST, Paris, France, 1976.

A biography of Issai Schur
http://www-history.mcs.st-andrews.ac.uk
/history/Mathematicians/Schur.html